THE ELEMENTS

OF THE

MECHANICS OF MATERIALS

AND OF

POWER TRANSMISSION

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PREFACE

This book is the result of an experience of some years in classroom work with engineering students, and is intended for use in technical schools and colleges.

It has frequently occurred to me that there is needless obscurity of statement in the average engineering text, with the consequent discouragement and retardation of young students. It has been my aim, therefore, to characterize the demonstrations in this text by completeness and simplicity of statement, in the belief that such treatment will greatly facilitate the study of more advanced works.

The Calculus has been introduced necessarily, but only in its elementary form, and chiefly in demonstrations and solutions. Such use of it will be, I believe, beneficial to the young student in showing him possibilities in the application of the subject to practical problems.

The book is in two parts. Part I is devoted to the elements of the Mechanics of Materials, and Part II to the elements of Power Transmission. It has been the aim to present both subjects only to the extent that will impart such a working knowledge of the fundamentals as will enable the student to grasp more extended works without aid.

Much of the matter on iron and steel in Chapter IX, Part I, has been taken by permission from Durand's "Practical Marine Engineering," and the chief sources of reference have been the works of Goodman and Unwin.

I am under obligation to my assistants in engineering, William L. De Baufre, Charles E. Conway, and Samuel P. Platt, — to the first two for valuable aid and suggestion and to the third for the care with which he made the tracings for the cuts.

WILLIAM R. KING.

Baltimore, *July* 4, 1911.

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PART I

THE ELEMENTS OF THE MECHANICS OF MATERIALS

CHAPTER I

MOMENTS. CENTER OF GRAVITY. MOMENT OF INERTIA. RADIUS OF GYRATION

1. **Introductory.** — The mechanics of materials, embracing the strength of materials, is an all-important subject to the engineering student. It includes all the calculations connected with the design of machines which admit of motion between some of their parts in the transmission of force, thus involving dynamical principles; and of the design of structures which remain in the static state of rest. It presupposes for the student a course in mechanics, but the questions of moments, center of gravity, moment of inertia, and radius of gyration are of such frequent application in mechanical design that a partial review of those subjects is given in this chapter.

2. **Moments.** — The moment of a force acting on a body may be defined as the tendency of the force to turn the body about a point, or about a fixed axis, and its measure is the product of the force by the perpendicular distance from the point, or from the axis, to the line of action of the force. The point, or axis, about which the moments are taken is called the *center of moments*, and the perpendicular distance

from the line of action of the force to the center of moments is called the *arm of the force.*

If there be a number of forces acting on the body, those tending to turn it in one direction may be regarded as positive and those tending to turn it in the opposite direction as negative. By common consent forces with a turning tendency in a clockwise direction are termed *positive* and those with a contraclockwise tendency are termed *negative.* It is immaterial which kind of force is termed positive and which negative, but having chosen one kind as positive in any investigation the choice must be adhered to and the opposite kind must be regarded as negative.

In Fig. 1 let the forces P, Q, and R act on the body in the directions indicated. If the body remains in equilibrium the underlying principle of moments asserts that *the algebraic sum of the moments of the forces about any point as a center, or about any line as an axis, must be zero, the point and the line being in the same plane.* In other words, the sum of the clockwise moments must equal the sum of the contraclockwise moments.

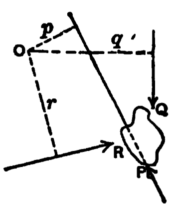


Fig. 1.

Let moments be taken about the point O. The force Q tends to turn the body in a clockwise direction about O and will be regarded as positive, and the tendencies of the forces P and R are to turn it in a contraclockwise direction about O and will be regarded as negative. The equation of moments will then be

$$Qq - Pp - Rr = 0.$$

This follows directly from the meaning of the word *equilibrium,* which implies that the body is at rest, and that condition can only result when there is no tendency to turn the body about the point O; that is, when the algebraic sum of the moments about O is zero. Should the line of action of a force pass

through the center of moments, the moment of that force would vanish.

In expressing the value of moments, the units of force, mass, area, and volume are placed first and the length units afterward. For example, a moment may be expressed as so many pounds-feet, and thus avoid confusion with work units.

3. **Center of Gravity.** — The center of gravity of a body or of a system of bodies is a point on which the body or system will balance in all positions, supposing the point to be supported, the body or system to be acted on only by gravity, and the parts of the body or system to be rigidly connected to the point.

It follows from this definition that all the particles of a body or of a system of bodies are acted on by a system of parallel forces, gravity acting on each, and that the algebraic sum of the moments of these forces about a line must be zero when the line passes through the center of gravity of the body or system; otherwise the body or system would not balance.

Let two heavy weights P and Q be situated as shown in Fig. 2. Join them with a straight line and divide it at C so that $\frac{P}{Q} = \frac{q}{p}$. If the weights be joined by a rigid rod without weight, the system will, by the principle of the lever, balance when supported at C. The center of gravity of the system is therefore at C. As the resultant of the weights is $P + Q$, the pressure on the support will be $P + Q$.

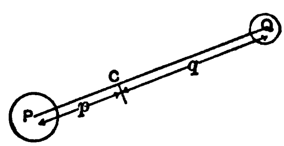


Fig. 2.

The center of gravity of a uniform straight rod is evidently at the middle point of its length, and when considering a body at rest we may assume its whole mass to be concentrated at its center of gravity.

Let the weights P and Q, Fig. 3, be attached to a balanced

rod as shown. Then, as we have just seen, the center of gravity will be so situated that $\frac{P}{Q} = \frac{n}{m}$, or $Pm = Qn$; that is, the algebraic sum of the moments is zero.

The distance of the center of gravity of a system from a point or from a line may readily be determined by moments. Thus, in Fig. 3, let $\bar{x}$ denote the distance of the center of gravity of the system from O, and let p and q be the distances,

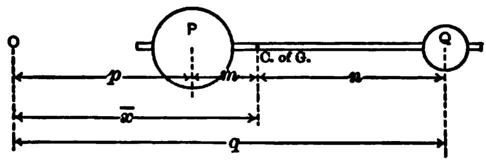


Fig. 3.

respectively, of the centers of gravity of P and Q from O. We have from the figure, $m = \bar{x} - p$ and $n = q - \bar{x}$. Since the center of gravity must be between P and Q, and since the turning tendency of P about the center of gravity is contra-clockwise and that of Q clockwise, we shall have by the principle of moments

$$P(\bar{x} - p) = Q(q - \bar{x}),$$

whence

$$\bar{x} = \frac{Pp + Qq}{P + Q}.$$

If the system be extended the result

$$\bar{x} = \frac{Pp + Qq + Rr + \text{etc.}}{P + Q + R + \text{etc.}},$$

will be obtained. This formula is extensively used in the solution of problems.

If a sheet of uniform thickness weighing M pounds per unit of area be considered, the weight of any area a will be Ma pounds,

and we may substitute Ma_1 for P and Ma_2 for Q and obtain

$$\bar{x} = \frac{Ma_1p + Ma_2q}{Ma_1 + Ma_2} = \frac{a_1p + a_2q}{a_1 + a_2} = \frac{a_1p + a_2q}{A},$$

in which A is the whole area. This may be expressed as follows:

The distance, $\bar{x}$, from O to the center of gravity is

$$\bar{x} = \frac{\text{Sum of the moments of all the elemental surfaces about } O}{\text{Whole surface}}$$

$$= \frac{\text{Moment of the whole surface about } O}{\text{Whole surface}}.$$

Similarly,

$$x = \frac{\text{Moment of the whole volume about } O}{\text{Whole volume}}.$$

The moment of the whole surface and of the whole volume is obtained by integration.

4. **The center of gravity** of a portion of the perimeter of a regular polygon or of an arc of a circle, considered as a thin wire, may be obtained thus:

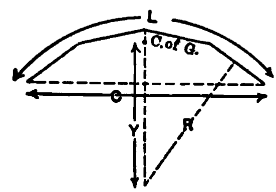


Fig. 4.

In Fig. 4, let L = the sum of the lengths of the sides of the portion of the perimeter taken, or the length of the arc in the case of a circle; R = the radius of the inscribed circle, or the radius of the circle; C = the chord of the arc of polygon or circle; Y = the distance of the center of gravity from the center of the circle. Then, in Fig. 5, let a denote the

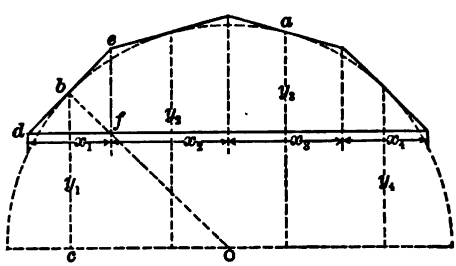


Fig. 5.

length of a side of the polygon. The center of gravity of each side will be at its middle point, and distant y_1, y_2, y_3, etc., from the diameter of the inscribed circle. Let x_1, x_2, x_3, etc., denote the projected lengths of the sides on the diameter. From the similar triangles *Obc* and *edf* we have

$$de : Ob = df : bc, \quad \text{or} \quad a : R = x_1 : y_1,$$

whence
$$y_1 = \frac{Rx_1}{a}.$$

Similarly,
$$y_2 = \frac{Rx_2}{a}, \text{ and so on.}$$

If w denotes the weight of each of the n sides of the polygon, we shall have

$$Y = \frac{Pp + Qq + \text{etc.}}{P + Q + \text{etc.}} = \frac{wy_1 + wy_2 + \text{etc.}}{nw} = \frac{y_1 + y_2 + \text{etc.}}{n}.$$

Substituting the values of y_1, y_2, y_3, etc., we have

$$Y = \frac{R}{na}(x_1 + x_2 + \text{etc.}).$$

But
$$na = L, \quad \text{and} \quad x_1 + x_2 + \text{etc.} = C.$$

Hence
$$Y = \frac{RC}{L}.$$

5. **The center of gravity** of a plane figure may be obtained graphically as follows:

Let *abcde*, Fig. 6, be any plane figure. Draw *eb* and *ec*. Let A_1 and g_1, A_2 and g_2, and A_3 and g_3 denote the areas and centers of gravity of the triangles *eab*, *ebc*, and *ecd* respectively. Join g_1 and g_2. The center of gravity of the figure *abce* must lie in this line. From g_1 lay off in any direction and to any scale the distance g_1t equal to A_2, and in the opposite direction and to the same scale lay off from g_2 the distance g_2s equal to A_1 and parallel to g_1t. Join *st*; then its intersection, *k*, with g_1g_2 is the center

of gravity of the figure *abce*. Join k and g_3. The center of gravity of $A_1 + A_2 + A_3$ will lie in this line. From k lay off in any direction a distance kr equal to the area A_3, and from g_3 lay off g_3v parallel to kr and equal to $A_1 + A_2$. Join vr, and its intersection with kg_3 gives G as the center of gravity of the whole figure.

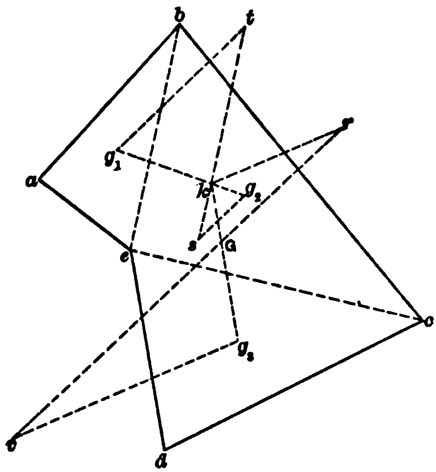


Fig. 6.

Should the surface contain a hole, as *fmn* in Fig. 7, we would proceed thus:

Denote by A_1 and g_1 and A_2 and g_2 the areas and centers of gravity of the whole figure *abcde* and of *fmn* respectively. Draw g_1g_2.

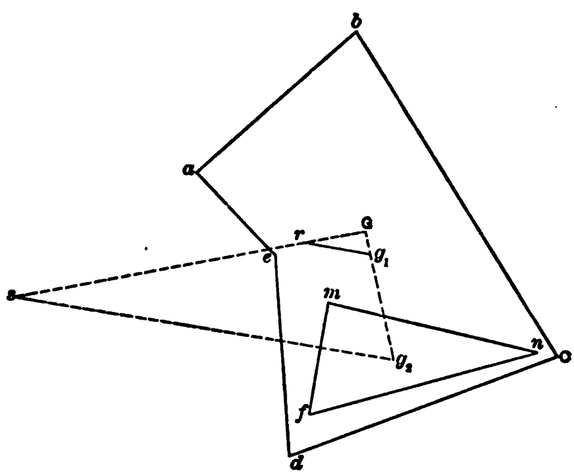


Fig. 7.

From g_2 lay off to scale, in any direction, g_2s equal to A_1, and parallel to it, and on the same side of g_1g_2, lay off g_1r. Join

sr and extend it to meet g_1g_2 produced at G. Then the point G is the required center of gravity.

The truth of these graphic methods may be shown as follows: Let O_1 and O_2, Fig. 8, be the positions of the centers of gravity of two areas A_1 and A_2 respectively, and let their common

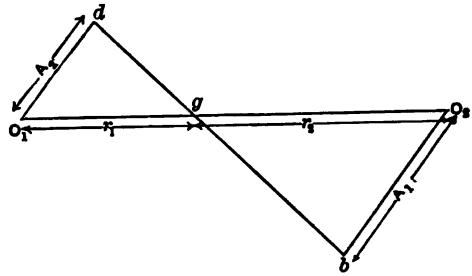


Fig. 8.

center of gravity be situated at g, distant r_1 from O_1 and r_2 from O_2. By the principle of moments we shall have $A_1r_1 = A_2r_2$. In any direction from O_2 lay off O_2b, whose length represents the area A_1 to some selected scale, and from O_1 lay off O_1d parallel to O_2b and of such length as to represent the area A_2 to the same scale. The triangles O_1dg and O_2bg are similar, and we have

$$O_2b : O_1d = O_2g : O_1g, \quad \text{or} \quad A_1 : A_2 = r_2 : r_1,$$

whence

$$A_1r_1 = A_2r_2.$$

That is, the line joining b and d passes through g. It will be observed that the lines O_2b and O_1d are laid off on opposite sides of the line joining O_1 and O_2 and at opposite ends of their respective areas and at any convenient angle. Should one of the areas, as A_2, be negative, *i.e.*, the area of a hole, or of a part cut out of the surface, then O_2b and O_1d must be laid off on the same side of O_1O_2, as shown in Fig. 9.

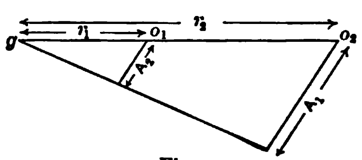


Fig. 9.

6. Applications. — To show the application of the foregoing principles to finding centers of gravity, the solutions of a few problems will be given.

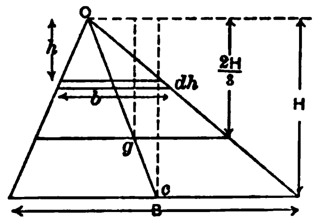


Fig. 10.

EXAMPLE I. — Find the center of gravity of a triangle.

Conceive the triangle, Fig. 10, to be divided into a great number of very narrow strips drawn parallel to one of the sides, as B. The center of gravity of each strip will be at its middle point, therefore the center of gravity of the triangle will lie in the locus of these middle points; that is, in the median Oc.

The elemental area of the triangle $= b \cdot dh$.

The moment of the elemental area $= h \cdot b \cdot dh$.

From similar triangles we have

$$b : B = h : H; \text{ whence, } b = \frac{Bh}{H}.$$

Then

$$\text{Moment of elemental area } = \frac{B \cdot h^2 \cdot dh}{H}.$$

$$\text{Moment of the whole area} = \frac{B}{H}\int_0^H h^2 dh = \frac{BH^2}{3}.$$

$$\text{The area of the triangle} = \frac{BH}{2}.$$

The distance, $\bar{x}$, of the center of gravity from the apex is

$$\bar{x} = \frac{\text{Moment of whole area}}{\text{Whole area}} = \frac{\frac{BH^2}{3}}{\frac{BH}{2}} = \frac{2H}{3}.$$

That is, the center of gravity is at a perpendicular distance below the apex equal to two-thirds of the altitude, and must

therefore be at g, the intersection of the median, Oc, and the parallel to the base distant $\frac{2H}{3}$ from the apex. From similar triangles it is seen that Og is two-thirds the length of the median, so that the center of gravity is on the median at two-thirds its length from the apex.

EXAMPLE II. — To find the center of gravity of a portion of a regular polygon or of a sector of a circle when considered as a lamina or thin sheet.

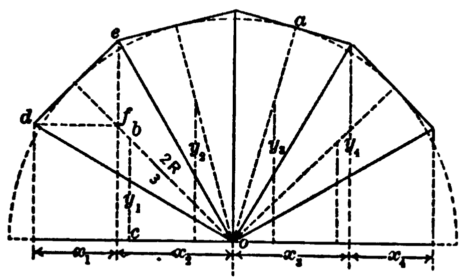


Fig. 11.

Referring to Fig. 11, and using the same notation as in Art. 4, we shall have $\frac{2R}{3}$ as the distance of the center of gravity of each of the triangles from the center of the inscribed circle, and they will be distant y_1, y_2, y_3, etc., from the diameter. The base of each of the triangles is a, and the projected lengths of these bases on the horizontal diameter are x_1, x_2, x_3, etc. From the similar triangles Obc and edf we have

$$\frac{2R}{3} : a = y_1 : x_1, \text{ whence } y_1 = \frac{2Rx_1}{3a};$$

similarly, $$y_2 = \frac{2Rx_2}{3a}, \text{ and so on.}$$

If w denotes the weight of each of the n triangles of the polygon we shall have

$$Y = \frac{wy_1 + wy_2 + \text{etc.}}{nw} = \frac{y_1 + y_2 + \text{etc.}}{n} = \frac{2R}{3na}(x_1 + x_2 + \text{etc.}) = \frac{2RC}{3L}.$$

EXAMPLE III. — A square is divided into four equal triangles by diagonals intersecting at O; if one triangle be removed, find the center of gravity of the figure formed by the three remaining triangles.

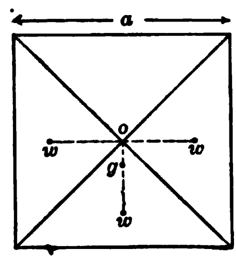


Fig. 12.

Let w denote the weight of each of the remaining triangles, and let a denote the side of the square, Fig. 12. The weight $2w$ of the side triangles may be supposed concentrated at their common center of gravity O. The distance of the center of gravity of the lower triangle from O is $\frac{a}{2} \times \frac{2}{3} = \frac{a}{3}$, by Example I. Then if $\bar{x}$ denotes the distance of the center of gravity of the three remaining triangles from O, we shall have

$$\bar{x} = \frac{Pp + Qq}{P + Q} = \frac{2w \times 0 + w \times \frac{a}{3}}{2w + w} = \frac{a}{9}.$$

That is, the required center of gravity is distant one-ninth the side of the square from the center of the square.

EXAMPLE IV. — A quarter of the area of a triangle is cut off by a line drawn parallel to the base. Find the center of gravity of the remaining quadrilateral.

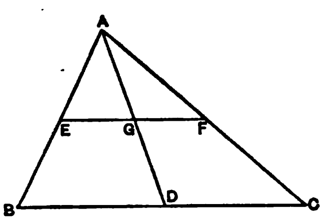


Fig. 13.

Let EF be parallel to the base BC, Fig. 13, and let the triangle AEF be the part cut off. The required center of gravity will lie in the median AD. The triangles AEF and ABC are similar, and are to each other in area as 1 : 4. Since the areas of similar triangles are to each other as the squares of homologous sides, we have

$$\overline{AG}^2 : \overline{AD}^2 = 1 : 4, \quad \text{whence} \quad AG = \frac{AD}{2}.$$

Let $\bar{x}$ denote the distance of the required center of gravity of the quadrilateral $BEFC$ from A. Then

$$\bar{x}=\frac{Pp-Qq}{P-Q}=\frac{4\times\frac{2\times AD}{3}-1\times\frac{AD}{2}\times\frac{2}{3}}{4-1}=\frac{AD\,(8-1)}{9}=\frac{7}{9}AD.$$

That is, the required center of gravity is in the median AD and at seven-ninths of its length from A.

EXAMPLE V. — Find the center of gravity of a cone.

Conceive the cone of Fig. 14 to be made up of a great number of thin sections, each parallel to the base. The center of gravity of each of these sections will be at its center, therefore the center of gravity of the cone will lie in the locus of the centers of these sections; that is, it will lie in the line joining the vertex with the center of gravity of the base.

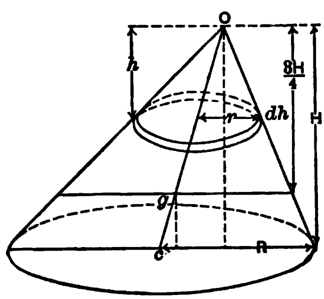


Fig. 14.

The elemental volume of the cone is $\pi r^2 dh$, and the moment of the elemental volume is $\pi r^2 h dh$. From similar triangles we have

$$R:r=H:h,\quad \text{whence } r=\frac{Rh}{H}.$$

Then by substitution we have

$$\text{Moment of elemental volume}=\frac{\pi R^2 h^3 dh}{H^2},$$

$$\text{Moment of whole volume}=\frac{\pi R^2}{H^2}\int_0^H h^3 dh=\frac{\pi R^2 H^2}{4}.$$

$$\text{Volume of cone}=\frac{\pi R^2 H}{3}.$$

If $\bar{x}$ denotes the distance of the center of gravity of the cone from the vertex, we shall have

$$\bar{x} = \frac{\text{Moment of whole volume}}{\text{Whole volume}} = \frac{\frac{\pi R^2 H^2}{4}}{\frac{\pi R^2 H}{3}} = \frac{3H}{4}.$$

That is, the center of gravity is at a perpendicular distance below the apex equal to three-fourths of the altitude, and must therefore be at g, the intersection of the line Oc joining the apex with the center of gravity of the base and the parallel to the base distant $\frac{3H}{4}$ from the apex. From similar triangles it is seen that $Og = \frac{3\overline{Oc}}{4}$, so that the center of gravity of the cone is in the line joining the vertex with the center of the base and at three-fourths its length from the vertex.

EXAMPLE VI. — Find the center of gravity of a semicircular arc, or wire.

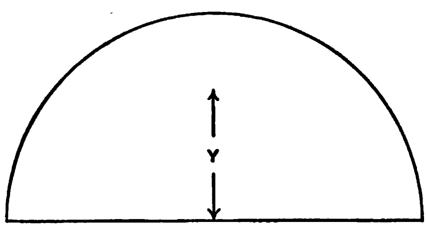


Fig. 15.

In Fig. 15 we have, from Art. 4, $Y = \frac{RC}{L}$, in which Y denotes the distance of the center of gravity from the center, L the length of the arc, and C the chord of the arc.

Then

$$Y = \frac{R \times 2R}{\pi R} = \frac{2R}{\pi} = \frac{D}{\pi}.$$

PROBLEMS

1. A rod 3 feet long and weighing 4 pounds has a weight of 2 pounds placed at one end; find the center of gravity of the system.

Ans. One foot from weighted end.

2. Find the center of gravity of a uniform circular disc out of which another circular disc has been cut, the latter being described on a radius of the former as a diameter.

Ans. One-sixth of radius of large circle from center.

3. A heavy bar 14 feet long is bent into a right angle so that the lengths of the portions which meet at the angle are 8 feet and 6 feet respectively; show that the distance of the center of gravity of the bar so bent from the point of the bar which was the center of gravity when the bar was straight, is $\frac{9\sqrt{2}}{7}$ feet.

4. The middle points of two adjacent sides of a square are joined and the triangle formed by this straight line and the edges is cut off; find the center of gravity of the remainder of the square.

Ans. $\frac{1}{21}$ of diagonal of square from center of square.

5. A piece of uniform wire is bent into the shape of an isosceles triangle; each of the equal sides is 5 feet long, and the other side is 8 feet long; find the center of gravity. *Ans.* $\frac{5}{18}$ of the altitude from the base.

6. Find the center of gravity of a figure consisting of an equilateral triangle and a square, the base of the triangle coinciding with one of the sides of the square.

Ans. At a distance from the base of the triangle equal to $\frac{3(4-\sqrt{3})}{26}$ times the base.

7. A table whose top is in the form of a right-angled isosceles triangle, the equal sides of which are 3 feet in length, is supported by three vertical legs placed at the corners; a weight of 20 pounds is placed on the table at a point distant 15 inches from each of the equal sides; find the resultant pressure on each leg. *Ans.* $8\frac{1}{3}$, $8\frac{1}{3}$, $3\frac{1}{3}$ lbs.

8. $ABCD$ is a quadrilateral figure such that the sides AB, AD, and the diagonal AC are equal, and also sides CB and CD are equal; find its center of gravity.

Ans. $\frac{2a^2+b^2}{6a}$ units from C, in which $a = AB$ and $b = CB$.

9. ABC represents a triangular board weighing 10 pounds. Suppose weights of 5 pounds, 5 pounds, and 10 pounds are placed at A, B, and C respectively. Where is the center of gravity of the whole?

Ans. At five-ninths of the median drawn from C.

10. A rod of uniform thickness is made up of equal lengths of three substances, the densities of which taken in order are in the proportion of 1, 2, and 3; find the position of the center of gravity of the rod.

Ans. At seven-eighteenths of the length of the rod from the end of the densest part.

11. Find the position of the center of gravity of a piece of wire bent to form three-fourths of the circumference of the circle of radius R.

Ans. On a line drawn from the center of the circle to a point bisecting the arc, and at a distance 0.3 R from the center.

12. A thin wire forms an arc of a circle the radius of which is 10 inches, and subtends an angle of 60°; find the distance of the center of gravity from the center. *Ans.* $\frac{30}{\pi}$ ins.

13. Find the position of the center of gravity of a balanced weight having the form of a circular sector of radius R, subtending an angle of 90°. *Ans.* 0.6 R from center of circle.

14. Find the center of gravity of a semicircular lamina, or sheet, of diameter D. *Ans.* $\frac{2D}{3\pi}$ from the center.

15. A square stands on a horizontal plane; if equal portions be removed from two opposite corners by straight lines parallel to a diagonal, find the least portion which can be left so as not to topple over.

Ans. Three-quarters of the area of the square.

16. Find the center of gravity of a trapezoid.

Ans. On the line joining the middle points of the bases, and at a perpendicular distance from the upper base equal to $\frac{H}{3} \cdot \frac{2B+b}{B+b}$, and from the lower base, $\frac{H}{3} \cdot \frac{2b+B}{B+b}$, B and b being the lower and upper bases respectively, and H the altitude. If the distance be measured on the line S joining the middle points of the bases, then $\frac{S}{3}$ must be substituted for $\frac{H}{3}$.

17. Find the center of gravity of a pyramid.

Ans. In the line joining the vertex with the center of gravity of the base and at three-fourths its length from the vertex.

18. Find the center of gravity of a frustum of a cone.

Ans. In the line joining the centers of gravity of the upper and lower bases, at a distance from the upper base equal to $\frac{H}{4} \cdot \frac{3R^2+2Rr+r^2}{R^2+Rr+r^2}$; and from the lower base $\frac{H}{4} \cdot \frac{3r^2+2Rr+R^2}{R^2+Rr+r^2}$, R and r being the radii of

the lower and upper bases respectively, and H the altitude. These results are true for the frustum of any pyramid by substituting B for R and b for r, B and b being homologous sides of the lower and upper bases respectively.

19. A lever safety valve is required to blow off at 70 pounds pressure per square inch. Diameter of valve, 3 inches; weight of valve, 3 pounds; short arm of lever, 2.5 inches; weight of lever, 11 pounds; distance of center of gravity of lever from fulcrum, 15 inches. Find the distance at which a cast-iron ball 6 inches in diameter must be placed from the fulcrum. The weight of the ball-hook is 0.6 pound, and that of a cubic inch of cast iron 0.26 lb. *Ans.* 35.48 ins.

20. A steel safety-valve lever 1 inch thick and 50 inches long tapers from 3 inches in depth at the fulcrum to 1 inch at the end. It overhangs the fulcrum 4 inches, the overhang having no taper. Diameter of valve, 4 inches; weight of valve, 4.75 pounds; short arm of lever, 3.5 inches. Find the distance from the fulcrum at which a cast-iron ball 9.5 inches in diameter must be placed in order that steam shall blow off at 125 pounds pressure per square inch. Weight of the ball-hook, 1.3 pounds; weight of a cubic inch of steel, 0.28 pound; weight of a cubic inch of cast iron, 0.26 pound. *Ans.* 42.35 ins.

21. A trapezoidal wall has a vertical back and a sloping front face; width of base, 10 feet; width of top, 7 feet; height, 30 feet. What horizontal force must be applied at a point 20 feet from the top in order to overturn it, *i.e.*, to make it pivot about the toe? Width of wall, 1 foot; weight of masonry in wall, 130 pounds per cubic foot.

Ans. 18,900 lbs.

22. Find the height of the center of gravity from the base of a column 4 feet square and 40 feet high, resting on a tapered base forming a frustum of a square-based pyramid 10 feet high and 8 feet square at the base.

Ans. 20.4 feet from base

Problems 23, 24, and 25 are to be solved graphically.

23. Find the position of the center of gravity of an unequally flanged section; top flange, 3 inches wide, 1.5 inches thick; bottom flange, 15 inches wide, 1.75 inches thick; web, 1.5 inches thick; total height, 18 inches.

Ans. 5.72 inches from the bottom edge.

24. Find the height of the center of gravity of a T-section from the foot, the top crosspiece being 12 inches wide and 4 inches deep; the stem, 3 feet deep and 3 inches wide.

Ans. 24.16 inches. (Check the result by seeing if the moments are equal about a line passing through the section at the height found.)

25. A square board weighs 4 pounds, and a weight of 2 pounds is placed at one of the corners. Find the position of the center of gravity of the board and weight.

Ans. On the diagonal drawn from the weighted corner and at two-thirds its length from the opposite corner.

7. Moment of Inertia. — The moment of inertia of a surface is the sum of the products of each elemental area of the surface by the square of its distance from an axis about which the surface is supposed to be revolving. If, instead of a surface, we have a body, then we must substitute the elemental volume for the elemental area in finding the moment of inertia. The moment of inertia varies according to the position of the axis, being smallest when the axis passes through the center of gravity.

It would be a crude and inaccurate method of finding the moment of inertia by actually dividing an area or a volume into its elements, multiplying each by the square of its distance from the axis, and then finally taking the sum of these products. We can, however, find the moment of inertia with accuracy by means of integration.

The strength of a beam or of a column depends upon the form as well as the area of its section, and in all calculations respecting the strength of beams and columns, the factor which gives expression to the effect of the form of the section is its moment of inertia about an axis passing through the center of gravity.

The moment of inertia of a surface is universally denoted by I, the axis being in the plane of the surface and passing through its center of gravity. When the axis is perpendicular to the plane of the surface, the moment of inertia is then termed the *polar moment* of inertia, and is denoted by I_p.

If we denote the product of a force, area, volume, or weight by its arm as the *first moment*, or simply the *moment*, of the force, area, volume, or weight, then we may conveniently denote the

product of the force, area, volume, or weight by the square of its arm as the *second moment* of the force, area, volume, or weight. Thus the moment of inertia is sometimes known as the *second moment.*

8. Relation between Moments of Inertia about Parallel Axes. — There is a relation between moments of inertia of a surface about parallel axes that is useful in the solution of problems.

Let I denote the moment of inertia of any surface about an axis through the center of gravity of the surface, I' the moment of inertia of the same surface about any other parallel axis, A the area of the surface, and D the perpendicular distance between the axes. Then we shall have $I' = I + AD^2$:

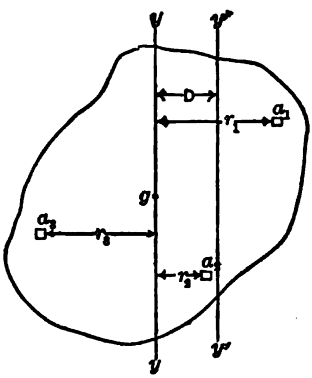


Fig. 16.

For, let yy, Fig. 16, be the axis through the center of gravity, and $y'y'$ the axis parallel to yy, both being in the plane of the surface. Denote the elemental areas of the surface by a_1, a_2, a_3, etc., and their distances from the axis yy by r_1, r_2, r_3, etc.

Then

$$\begin{aligned} I' &= a_1(r_1 - D)^2 + a_2(D - r_2)^2 + a_3(D + r_3)^2 + \text{etc.} \\ &= a_1(r_1^2 - 2r_1D + D^2) + a_2(D^2 - 2r_2D + r_2^2) \\ &\qquad + a_3(D^2 + 2r_3D + r_3^2) + \text{etc.} \\ &= (a_1r_1^2 + a_2r_2^2 + a_3r_3^2) + (a_1 + a_2 + a_3)D^2 \\ &\qquad - 2D(a_1r_1 + a_2r_2 - a_3r_3). \end{aligned}$$

The first term of the second member of this equation is, by our definition, the moment of inertia, I, of the surface about the axis yy through the center of gravity; the second term

becomes AD^2, in which A denotes the whole area; and the third term will reduce to zero, because the quantity within the parenthesis is the algebraic sum of the moments of the elemental areas about a line passing through their center of gravity. Therefore $$I' = I + AD^2.$$

9. **Suppose an axis perpendicular** to the plane of the surface represented in Fig. 17 to pass through O. Let a be an elemental area of the surface, distant r from O, and let X and Y be rectangular axes passing through O and lying in the plane of the surface. Then the polar moment of a is $I_p = ar^2$. The moment of inertia of a about X is $I_x = ay^2$, and similarly $I_y = ax^2$. But we have $x^2 + y^2 = r^2$, therefore $ax^2 + ay^2 = ar^2$. That is, the polar moment of inertia of any surface is equal to the sum of the moments of inertia of the surface about any two rectangular axes lying in the plane of the surface and passing through the polar axis of revolution; that is,

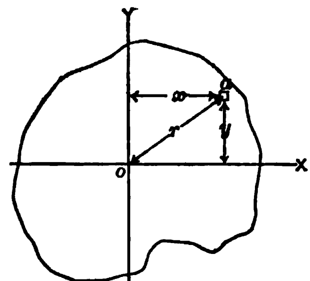


Fig. 17.

$$I_p = I_x + I_y.$$

10. **Radius of Gyration.** — We have seen, Art. 8, that $I = a_1r_1^2 + a_2r_2^2 + a_3r_3^2 + \text{etc.}$, in which $a_1 + a_2 + a_3 + \text{etc.} = A$, the whole area. If we can conceive the whole area to be condensed into a single particle, distant K from the axis of rotation, we shall have $I = AK^2$. This imaginary point at which the particle is supposed to be situated is known as the center of gyration, and its distance, K, from the axis is the radius of gyration. We have then, $K = \sqrt{\frac{I}{A}}$, in which the mass, M, the volume, V, or the weight, W, may be substituted for the area, A.

11. **Illustrations.** — To illustrate the methods of finding moments of inertia and radii of gyration a few examples follow.

EXAMPLE I. — To find the least moment of inertia and least radius of gyration of the surface of a parallelogram.

The least moment of inertia is that about an axis passing through the center of gravity and parallel to the bases. Let XX, Fig. 18, be the axis through the center of gravity and parallel to the base B. It bisects the altitude h.

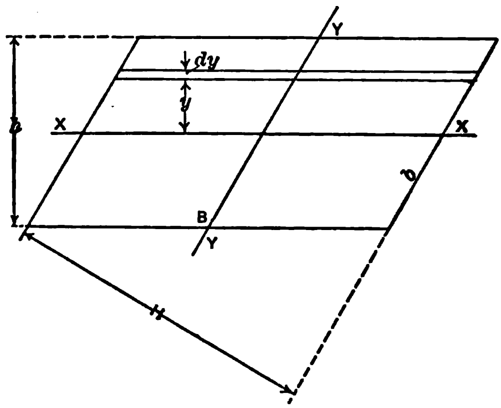


Fig. 18.

Suppose the surface to be divided into an infinite number of strips parallel to the axis, each of thickness dy. Consider one of these strips distant y from the axis.

Then $$\text{Elemental area} = Bdy,$$

$$\text{Second moment of elemental area} = dI = By^2dy.$$

Then

$$I = B\int_{-\frac{h}{2}}^{\frac{h}{2}} y^2dy = B\left[\frac{y^3}{3}\right]_{-\frac{h}{2}}^{\frac{h}{2}} = B\left[\frac{h^3}{24} - \left(-\frac{h^3}{24}\right)\right] = \frac{Bh^3}{12}.$$

If the axis YY parallel to the base b be taken, we shall have

$$I = \frac{bH^3}{12}.$$

For the radius of gyration we shall have

$$K = \sqrt{\frac{I}{A}} = \sqrt{\frac{Bh^3}{12\,Bh}} = \frac{h}{6}\sqrt{3} \text{ for the axis } XX,$$

and

$$K = \sqrt{\frac{bH^3}{12\,bH}} = \frac{H}{6}\sqrt{3} \text{ for the axis } YY.$$

If the moment of inertia with respect to an axis coinciding with the base be desired, we have $I' = I + AD^2$, in which D denotes the perpendicular distance between the axes, denoted in this instance by $\frac{h}{2}$ when B is the axis, and by $\frac{H}{2}$ when b is the axis.

Then

$$I' = \frac{Bh^3}{12} + Bh \times \frac{h^2}{4} = \frac{Bh^3}{3} \text{ for the base } B \text{ as axis,}$$

and

$$I' = \frac{bH^3}{12} + bH \times \frac{H^2}{4} = \frac{bH^3}{3} \text{ for the base } b \text{ as axis.}$$

Also

$$K' = \sqrt{\frac{Bh^3}{3\,Bh}} = \frac{h}{3}\sqrt{3} \text{ for the base } B \text{ as axis,}$$

and

$$K' = \sqrt{\frac{bH^3}{3\,bH}} = \frac{H}{3}\sqrt{3} \text{ for the base } b \text{ as axis.}$$

EXAMPLE II. — Find the moment of inertia and radius of gyration of a circular surface about an axis passing through the center and perpendicular to the plane of the surface.

The axis being perpendicular to the surface it is the polar moment of inertia that is required.

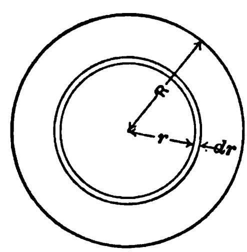


Fig. 19

Conceive the circular surface of Fig. 19 to be made up of an infinite number of concentric circular strips, each of thickness dr. Consider the strip distant r from the center.

Then Elemental area $= 2\,\pi r dr$,

Second moment of elemental area $= dI = 2\,\pi r^2 r dr$.

Then
$$I = 2\pi\int_0^R r^3dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}.$$

$$K = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi D^4}{8\pi D^2}} = \frac{D\sqrt{2}}{4} = \frac{R\sqrt{2}}{2}.$$

EXAMPLE III. — Find the polar moment of inertia and radius of gyration of a right circular cone about its axis.

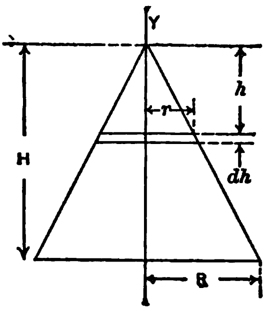


Fig. 20.

Conceive the volume of the cone, Fig. 20, to be made up of an infinite number of circular strips, each of thickness dh. Consider the strip of radius r and distant h from the vertex.

Then

$$\text{Elemental volume} = \pi r^2 dh.$$

If we imagine the elemental volume to be condensed into a single particle, the distance of this particle from the axis will be the radius of gyration K, which for the circle is $\frac{r\sqrt{2}}{2}$.

Hence,

$$\text{Second moment of elemental volume} = dI_p = \pi r^2 K^2 dh = \frac{\pi r^4 dh}{2}.$$

Then
$$I_p = \frac{\pi}{2}\int_0^R r^4 dh.$$

By similar triangles, $\frac{h}{H} = \frac{r}{R}$, whence $h = \frac{Hr}{R}$, and $dh = \frac{Hdr}{R}$.

Then
$$I_p = \frac{\pi H}{2R}\int_0^R r^4 dr = \frac{\pi R^4 H}{10}.$$

$$K = \sqrt{\frac{I_p}{V}} = \sqrt{\frac{\frac{\pi R^4 H}{10}}{\frac{\pi R^2 H}{3}}} = \frac{R\sqrt{30}}{10}.$$

PROBLEMS

Find the moment of inertia and radius of gyration of the following:

1. Triangle about an axis through vertex and parallel to base.

$$Ans.\ I' = \frac{BH^3}{4};\ K = \frac{H}{2}\sqrt{2}.$$

2. Triangle about an axis through the center of gravity and parallel to base.

$$Ans.\ I = \frac{BH^3}{36};\ K = \frac{H}{18}\sqrt{18}.$$

3. Triangle about an axis coinciding with base.

$$Ans.\ I' = \frac{BH^3}{12};\ K = \frac{H}{6}\sqrt{6}.$$

4. Trapezoid about an axis coinciding with small base.

$$Ans.\ I' = \frac{H^3(3B+b)}{12};\ K = H\sqrt{\frac{3B+b}{6(B+b)}}.$$

5. Trapezoid about an axis coinciding with large base.

$$Ans.\ I' = \frac{H^3(3b+B)}{12};\ K = H\sqrt{\frac{3b+B}{6(B+b)}}.$$

6. Trapezoid about an axis through center of gravity and parallel to base.

$$Ans.\ I = \frac{H^3(B^2+4Bb+b^2)}{36(B+b)};\ K = \frac{H}{6(B+b)}\sqrt{2(B^2+4Bb+b^2)}.$$

7. Square about its diagonal.

$$Ans.\ I = \frac{a^4}{12};\ K = \frac{a}{12}\sqrt{12}.$$

8. Circle about a diameter.

$$Ans.\ I = \frac{\pi D^4}{64};\ K = \frac{D}{4}.$$

9. Hollow circle about a diameter.

$$Ans.\ I = \frac{\pi}{64}(D^4 - d^4);\ K = \frac{\sqrt{D^2+d^2}}{4}.$$

Find the polar moment of inertia and radius of gyration of the following named surfaces:

10. Parallelogram about a pole passing through its center of gravity.

$$Ans.\ I_p = \frac{BH}{12}(H^2+B^2);\ K = \sqrt{\frac{H^2+B^2}{12}}.$$

11. Hollow circle about a pole passing through center.

$$Ans.\ I_p = \frac{\pi}{32}(D^4 - d^4);\ K = \sqrt{\frac{D^2+d^2}{8}}.$$

Find the polar moment of inertia and radius of gyration of the following named solids:

12. Cylinder about a pole coinciding with its axis.

$$Ans.\ I_p = \frac{\pi D^4 H}{32};\ K = \frac{D\sqrt{2}}{4}.$$

13. Hollow cylinder about a pole coinciding with its axis.

$$Ans.\ I_p = \frac{\pi H}{32}(D^4 - d^4);\ K = \sqrt{\frac{D^2 + d^2}{8}}.$$

14. Sphere about a diameter

$$Ans.\ I_p = \frac{8\pi R^5}{15};\ K = \frac{R\sqrt{10}}{5}.$$

15. A bar of rectangular section about a pole passing through the center of figure.

$$Ans.\ I_p = \frac{LBH}{12}(L^2 + B^2);\ K = \sqrt{\frac{L^2 + B^2}{12}}.$$

CHAPTER II

BENDING MOMENT. BENDING MOMENT DIAGRAM. SHEAR. SHEAR DIAGRAM.

12. **Bending Moments.** — When forces act on a body in such a manner as to tend to give it a spin or a rotation about an axis without any tendency to shift its center of gravity, the body is said to be acted on by a couple. A couple consists of two parallel forces of equal magnitude acting in opposite directions but not in the same line, the arm of the couple being the perpendicular distance between the lines of action of the forces.

A beam is subjected to a bending moment when it is so acted upon at its ends by equal and opposite couples that there is a tendency to turn it in opposite directions.

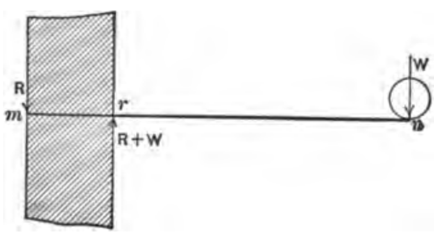


Fig. 21.

Thus, the beam *mn* of Fig. 21 is acted on by the equal and opposite couples, R and R, and W and W, the tendency being to turn the beam in opposite directions about the point r; that is, to bend it at r. In Fig. 22, the couples whose moments are $R_1 \times mr$ and $R_2 \times nr$ have the same effect on the beam as those of Fig. 21. The beam of Fig. 21 is called a *cantilever*, from the

nature of its support, while the beam of Fig. 22 is called a simple beam. The forces R, R_1 and R_2 are the support reactions.

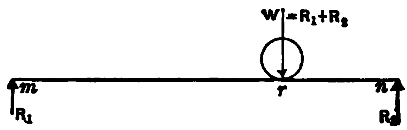


Fig. 22.

13. General Case of Bending Moments. — *The bending moment at any section of a beam is the algebraic sum of the moments of all the external forces acting to the left of the section.* It is assumed that forces acting upward are positive, and those acting downward are negative, so that bending moments may be positive or negative.

In graphical constructions the signs of bending moments are of the first importance and are determined by the following rule:

Bending moments which tend to bend a beam or cantilever concave upward, ⌣, are regarded as positive, and when they tend to bend in the reverse way, ⌢, they are negative.

It should be observed that it is merely to avoid confusion in the construction of diagrams that the external forces to the left of the section were considered when defining the bending moment at any section of a beam. Moments may equally well be taken to the right of the section and the same value be obtained for the bending moment at the section. When bending moments are obtained by calculation rather than by construction, the side involving the least calculation in taking moments should be selected, though the calculations for both sides afford a positive check as to the accuracy of the work. To avoid confusion with work units, bending moments are expressed in pounds-inches, pounds-feet, or tons-inches, as may be found most convenient.

The beam of Fig. 23 is supposed to be without weight. The bending moment at section E is, taking moments to the left of E,

$$M = R_1 \times aE - W_1 \times qE - W_2 \times pE, \qquad (1)$$

or, taking moments to the right of E,

$$M = R_2 \times bE - W_3 \times rE. \qquad (2)$$

Suppose the distances and weights to be as shown in the figure. Before finding the bending moments we must first find the reactions, R_1 and R_2.

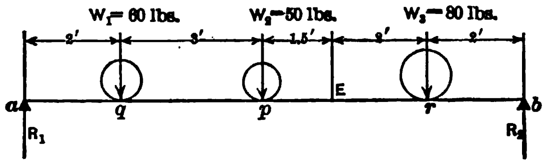


Fig. 23.

Taking moments about the left support, we have

$$R_2 \times 10.5 = 80 \times 8.5 + 50 \times 5 + 60 \times 2,$$

whence $$R_2 = 100 \text{ pounds.}$$

Taking moments about the right support, we have

$$R_1 \times 10.5 = 60 \times 8.5 + 50 \times 5.5 + 80 \times 2,$$

whence $$R_1 = 90 \text{ pounds,}$$

which might have been expected, since $R_1 + R_2$ should equal $W_1 + W_2 + W_3$.

By substitution in equation (1), we have for the bending moment at E,

$$M = 90 \times 6.5 - 60 \times 4.5 - 50 \times 1.5 = 240 \text{ pounds-feet.}$$

Substituting in equation (2),

$$M = 100 \times 4 - 80 \times 2 = 240 \text{ pounds-feet.}$$

It is thus seen that the bending moment at a section is the same, whether the moments be taken to the left or to the right of the

section. In this instance the taking of moments to the right involved the least calculation.

14. Bending Moment Diagrams. — A diagram may be made to show graphically the bending moment at any section of a beam. Such diagrams are known as bending-moment diagrams.

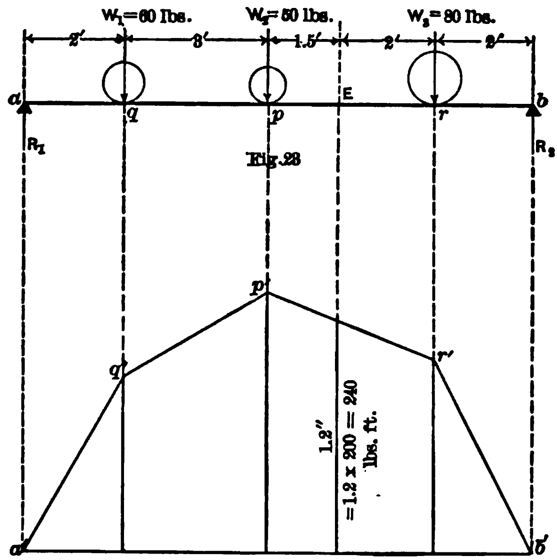


Fig. 24.

For example, take the beam of Fig. 23. The bending moment at q is

$$M = R_1 \times 2 = 90 \times 2 = 180 \text{ lbs.-ft.};$$

at p $\quad M = R_1 \times 5 - W_1 \times 3 = 90 \times 5 - 60 \times 3 = 270$ lbs.-ft.;

at r $\quad M = R_1 \times 8.5 - W_1 \times 6.5 - W_2 \times 3.5$

$$= 90 \times 8.5 - 60 \times 6.5 - 50 \times 3.5 = 200 \text{ lbs.-ft.}$$

If, on a base line $a'b'$, Fig. 24, and to a scale of 1 inch = 200 pounds-feet, we erect ordinates to represent these bending moments, and then join their extremities with the broken line $a'q'p'r'b'$, the inclosed figure is a diagram whose ordinates

beneath any section of the beam will measure the bending moment at the section to the scale adopted. Thus the ordinate for the bending moment at q measures $\frac{180}{200} = 0.9$ inch; that at p, $\frac{270}{200} = 1.35$ inches; and that at r, $\frac{200}{200} = 1$ inch. The ordinate beneath E measures 1.2 inches, which, to scale, represents a bending moment of $1.2 \times 200 = 240$ pounds-feet, as already found by calculation.

15. Shear. — Shearing stresses exist when couples, acting like a pair of shears, tend to cut a body between them. Beams acted on by couples are subjected to shearing stresses as well as to bending moments, the latter being far more important in beams of lengths ordinarily encountered. The failure of very short beams will be invariably from shear.

The shear at any section of a beam or of a cantilever is equal to the algebraic sum of the forces to the left of the section, the upward forces being regarded as positive and the downward negative.

16. Shear Diagrams. — Diagrams made to show graphically the amount of the shear at any section of a beam are known as shear diagrams. In their construction, attention must be paid to signs, so that in cases where the shear is partly positive and partly negative the positive part may be placed above the shear axis, or base line, and the negative part below.

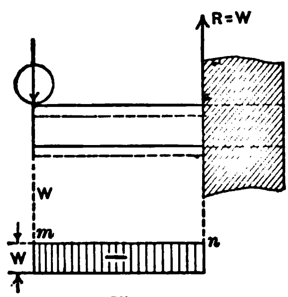


Fig. 25.

The failure from overload of a short cantilever, Fig. 25, will be from shear, the projecting part shearing off bodily from the part built in. According to our definition the shear at all sections of the beam is constant and equal to $-W$. The shear is therefore represented graphically by the small rectangular shaded diagram of Fig. 25, each of the ordinates of which is equal to $-W$. As the shear diagram is negative, it is placed below the shear axis mn.

In the case of the simple beam of Fig. 26, a failure from shear will occasion the part between the supports to slide down, as shown by the dotted lines. By our definition the shear at any section to the left of the load W is R_1, and at any section between W and the right support it is $R_1 - W = - R_2$, since $R_1 + R_2 = W$.

The shaded part of Fig. 26 represents the shear, and shows that it changed sign under the load, the positive part being placed above the shear axis mn and the negative part below.

As in the case when defining bending moments, only the external forces to the left of the section were considered when defining the shear at the section, but this, as in the case of bending moments, was only for convenience. If W_s denotes the sum of all the loads between the left support and a given section, and W_s' the sum of all the loads between the section and the right support, then evidently $R_1 + R_2 = W_s + W_s'$, whence $R_1 - W_s = - (R_2 - W_s')$. The first member of this equation is the algebraic sum of the forces to the left of the given section and is the shear at the section, and the second member is the negative of the algebraic sum of the forces to the right of the section. The shear at a given section is, then, the algebraic sum of the external forces to the right or to the left of the section, but with contrary signs.

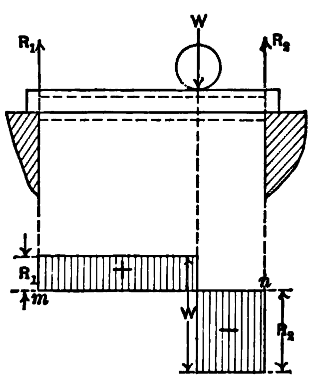


Fig. 26.

Let the beam of Fig. 27 be loaded with W_1, W_2, W_3 at distances d_1, d_2, d_3 respectively from any given section a. Denote the support reactions by R_1 and R_2, and their distances from a by r_1 and r_2 respectively.

To construct the shear diagram of the beam we select a base

line, or axis, *mn*, and having determined to an appropriate scale the values of R_1 and R_2 from the given weights W_1, W_2, and W_3, we proceed as follows:

By our definition the shear at any section to the left of W_1 is R_1, which gives the point *b* of Fig. 28. Immediately the

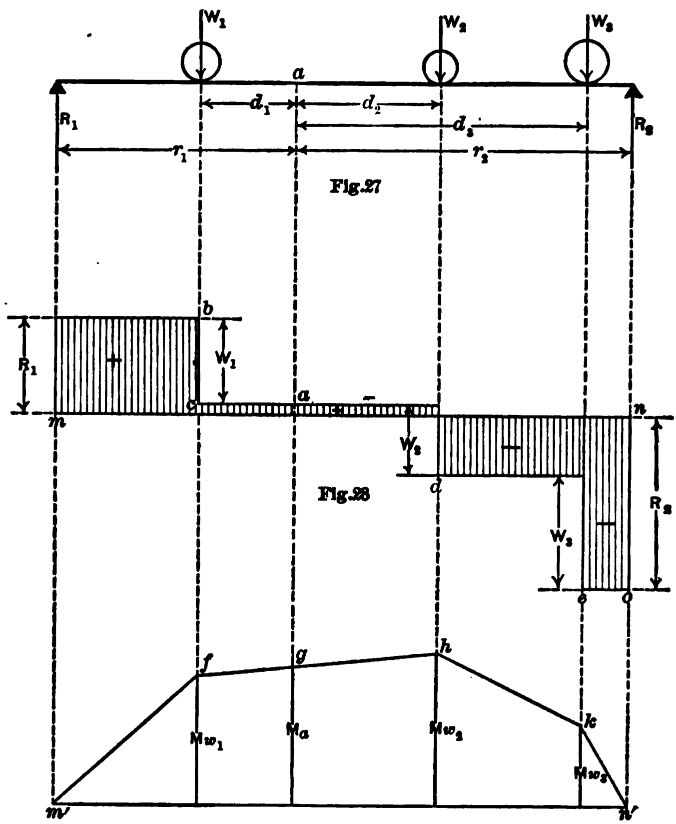


Fig. 29.

point of application of W_1 is passed the shear becomes $R_1 - W_1$, giving the point *c*, and this shear continues unchanged up to the point of application of W_2, but immediately this point is passed the shear becomes $R_1 - W_1 - W_2$, giving the point *d*,

the shear becoming negative. This shear continues unchanged until the point of application of W_3 is passed, when the shear becomes $R_1 - W_1 - W_2 - W_3 = - R_2$, giving the point e. This shear continues unchanged to the right support, giving the point o and completing the diagram.

Figures 27 and 28 were constructed to the following scales: Linear, 1 inch = 4 feet; load, 1 inch = 150 pounds.

The data of the beam and loads were: Length of beam = 12 feet = 3 inches to scale; d_1 = 2 feet = 0.5 inch to scale; d_2 = 3 feet = 0.75 inch to scale; d_3 = 6 feet = 1.5 inches to scale; r_1 = 5 feet = 1.25 inches to scale; r_2 = 7 feet = 1.75 inches to scale; W_1 = 60 pounds = 0.4 inch to scale; W_2 = 45 pounds = 0.3 inch to scale; W_3 = 90 pounds = 0.6 inch to scale.

The support reactions were first found as follows:

Taking moments about the left support, we have

$$R_2 (r_1 + r_2) = W_3 (r_1 + d_3) + W_2 (r_1 + d_2) + W_1 (r_1 - d_1),$$

or $$12 R_2 = 90 \times 11 + 45 \times 8 + 60 \times 3,$$

whence $$R_2 = 127.5 \text{ pounds}.$$

Therefore $$R_1 = 60 + 45 + 90 - 127.5 = 67.5 \text{ pounds}.$$

Reduced to scale the reactions R_1 and R_2 measure

$$\frac{67.5}{150} = 0.45 \text{ inch and } \frac{127.5}{150} = 0.85 \text{ inch respectively.}$$

The bending-moment diagram, Fig. 29, was constructed on the base line $m'n'$ to a scale such that 1 inch in depth represents 300 pounds-feet. The bending moments at W_1, W_2, and W_3 were calculated as follows:

$$M_{W_1} = R_1 (r_1 - d_1) = 67.5 \times 3 = 202.5 \text{ pounds-feet} = \frac{202.5}{300}$$
$$= 0.675 \text{ inch to scale.}$$

$$M_{W_2} = R_1 (r_1 + d_2) - W_1 (d_1 + d_2) = 67.5 \times 8 - 60 \times 5$$
$$= 240 \text{ pounds-feet.} = \frac{240}{300} = 0.8 \text{ inch to scale.}$$

$$M_{W_3} = R_1(r_1 + d_3) - W_1(d_1 + d_3) - W_2(d_3 - d_2)$$
$$= 67.5 \times 11 - 60 \times 8 - 45 \times 3 = 127.5 \text{ pounds-feet.}$$
$$= \frac{127.5}{300} = 0.425 \text{ inch to scale.}$$

Ordinates equal in length to the scale measurements representing these bending moments were erected on $m'n'$, as shown, and their extremities joined by the broken line $m'fghkn'$. The ordinate of the resulting diagram beneath any section of the beam is the scale measurement of the bending moment at the section. Thus the ordinate under the section at a measures $\frac{29}{40}$ of an inch, corresponding to a bending moment of $\frac{29}{40} \times 300 = 217.5$ pounds-feet.

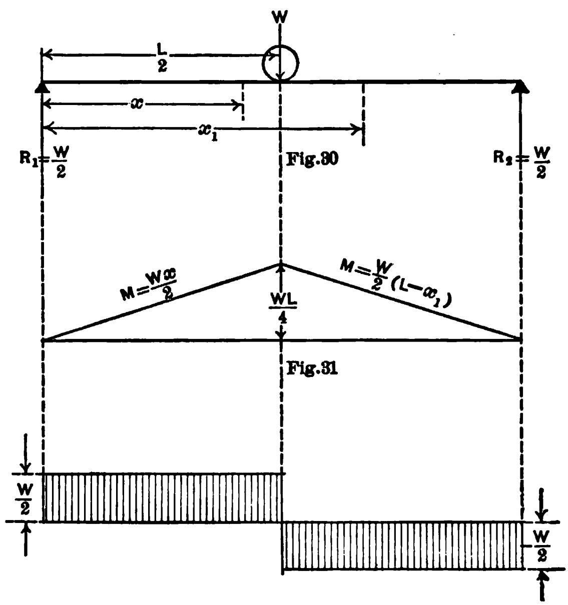


Fig. 32.

17. Simple Beam with Concentrated Load at Middle. — The beam of Fig. 30 is supported at the ends and has a load W concentrated at its middle.

The reaction at each support is $\frac{W}{2}$. The bending moment at any section between the left support and the middle and distant x from the support is $M = \frac{Wx}{2}$, and increases directly as x increases. At the middle, $x = \frac{L}{2}$, and $M = \frac{WL}{4}$. At any section between the middle and the right support and distant x_1 from the left support,

$$M = \frac{Wx_1}{2} - W\left(x_1 - \frac{L}{2}\right) = \frac{Wx_1}{2} - Wx_1 + \frac{WL}{2} = \frac{W}{2}(L - x_1),$$

and decreases as x increases, becoming 0 when $x_1 = L$. The maximum bending moment is therefore at the middle and is equal to $\frac{WL}{4}$. Since the equations $M = \frac{Wx}{2}$ and $M = \frac{W}{2}(L - x_1)$ are those of straight lines, the bending-moment diagram has the triangular form shown in Fig. 31.

Commencing at the left support the shear is $\frac{W}{2}$ and remains unchanged until directly after passing the middle section, when it becomes $\frac{W}{2} - W = -\frac{W}{2}$, changing sign at the middle section. The shear diagram is then as shown in Fig. 32.

18. Simple Beam Uniformly Loaded. — The beam of Fig. 33 is the same as that of Fig. 30, but the load W, instead of being concentrated at the middle, is uniformly distributed over the whole length of the beam with w pounds per unit of length, so that wL, the whole load, is equal to W.

It is evident that the support reactions are each $\frac{wL}{2}$ pounds. To find the bending moment at any section a, distant x from the left support, we may assume the uniform load to be made up of a number of parallel forces each equal to w. There are

wx of these forces between section a and the left support, and we may substitute for them their resultant, wx, acting at their center of gravity, which is midway between the section and the support.

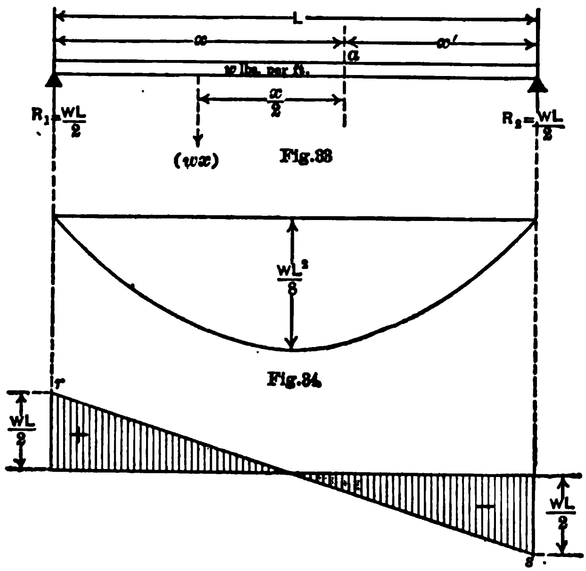


Fig. 35.

The bending moment at the section a is then

$$M_a = R_1 x - wx \times \frac{x}{2} = \frac{wLx}{2} - \frac{wx^2}{2} = \frac{wx}{2}(L - x) = \frac{wxx'}{2}.$$

That is, the bending moment at the section is proportional to the product of the segments into which the section divides the beam, and the bending-moment diagram is therefore a parabola, as shown in Fig. 34, having its axis vertical and under the middle of the beam; for it is a property of the parabola that if a diameter be drawn to intersect a chord the product of the segments of the chord is proportional to the length of that part of the diam-

eter included between its vertex and its point of intersection with the chord.

The maximum bending moment is evidently at the middle of the beam, so if in the general expression $M_a = \frac{wLx}{2} - \frac{wx^2}{2}$ for the bending moment we let $x = \frac{L}{2}$, we get

$$M_{\max} = \frac{wL^2}{4} - \frac{wL^2}{8} = \frac{wL^2}{8} = \frac{WL}{8},$$

in which $W = wL$ is the whole weight. A comparison of this result with that obtained for the maximum-bending moment in Art. 17 discloses the fact that a load concentrated at the middle of a simple beam occasions a bending moment twice as great as that due to the same load uniformly distributed.

The shear diagram of Fig. 35 may readily be constructed. Denoting the shear by V, we shall have for any section distant x from the left support,

$$V = R_1 - wx = \frac{wL}{2} - wx,$$

which is the equation of a straight line, the origin being at the left support. It is seen that V has its maximum value, $\frac{wL}{2}$, when $x = 0$, and that it decreases as x increases until, when $x = \frac{L}{2}$, it becomes 0. For values of x greater than $\frac{L}{2}$ the value of V becomes negative, and when $x = L$ the value of V becomes $-\frac{wL}{2}$. The straight line whose equation is $V = \frac{wL}{2} - wx$ is therefore *rs*.

19. Dangerous Section. — The maximum bending moment is the controlling influence in the design of beams, and the section at which it occurs is known as the *dangerous section.*

It has been shown, Art. 18, that in the case of a uniformly loaded beam the bending-moment curve is parabolic, and that

therefore the ordinate representing the bending moment is a continuous function of x. The determination of the dangerous section would then be to find that value of x which would make the bending moment a maximum. To do this we would place the first x-derivative of the bending moment equal to zero, and the resulting value of x would be the abscissa of the dangerous section.

For example, the general expression for the bending moment of the uniformly loaded beam of Fig. 33 is

$$M = R_1 x - \frac{wx^2}{2}.$$

Then
$$\frac{dM}{dx} = R_1 - wx,$$

$$R_1 - wx = 0, \quad \text{whence} \quad x = \frac{R_1}{w} = \frac{wL}{2w} = \frac{L}{2}.$$

The maximum bending moment in this case, and therefore the dangerous section, is at the middle of the beam.

If, in the general expression $\frac{wL}{2} - wx$ for the shear in Art. 18, we substitute for x the value $\frac{L}{2}$, the result is zero, from which we infer that in the case of uniform loading the maximum bending moment occurs at the section where the shear is zero.

In the equation $\frac{dM}{dx} = R_1 - wx$ it will be observed that the second member is the general expression for the shear over the whole beam, from which we may also infer that *the first derivative of the bending moment is the shear.*

If, however, in addition to the uniformly distributed load the beam were subjected to one or more concentrated loads, or if there were no uniform load and the beam were subjected simply to one or more concentrated loads, the ordinate representing the bending moment would no longer be a continuous function of

x. The first derivative of the bending moment at any section would, however, still be the shear at that section; but the shear at the dangerous section might not be zero, because the shear is not necessarily zero at any section. If the construction of the shear diagram shows that the shear is not zero at any section, it will also show that at some one or more sections the shear suddenly changed sign, or passed through zero, and the maximum bending moment will be found to occur at one of these sections.

20. Simple Beam with two Equal and Symmetrically Placed Loads. — The beam of Fig. 36 is supported at the ends and has two equal and symmetrically placed loads.

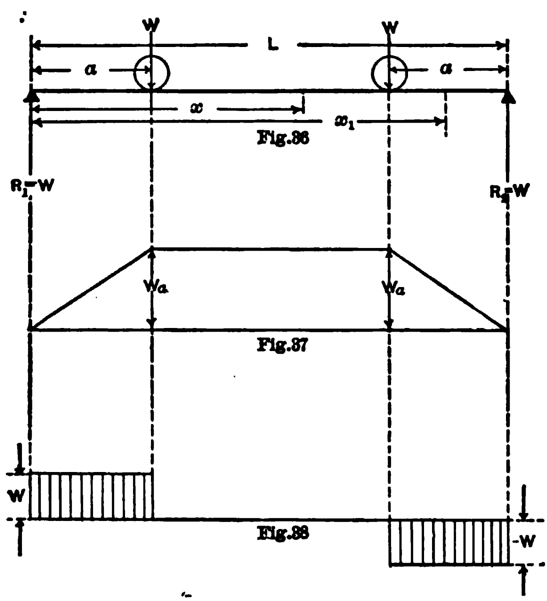


Fig. 36
Fig. 37
Fig. 38

At the section under the load nearest the left support, $M = Wa$; at any section between the loads and distant x from the left support the bending moment is $M_x = Wx - W(x - a) = Wa$; at any section between the second load and the right support

and distant x_1 from the left support the bending moment is $M_{x_1} = Wx_1 - W(x_1 - a) - W[x_1 - (L - a)] = WL - Wx_1$. For $x_1 = L$ in the last equation we get $M_{x_1} = 0$. From these expressions for the bending moments the diagram of Fig. 37 was constructed.

Commencing at the left support the shear is W and is unchanged until the first load is passed, when it becomes $W - W = 0$. There is no further change until the second load is passed, when the shear becomes $W - W - W = -W$, and so continues to the right support. The shear diagram is, then, as shown in Fig. 38.

21. Bending-Moment and Shear Diagrams of Cantilevers. — The cantilever of Fig. 39 has a concentrated load W at the end, a concentrated load W_1 at an intermediate point between the end and the support, and a uniformly distributed load of w pounds per foot over a portion of its length. The construction of the bending-moment and shear diagrams of this beam will serve to illustrate the three cases of a cantilever: (*a*) Loaded at the end. (*b*) A concentrated load at some point between the end and the support. (*c*) Loaded uniformly with w pounds per unit of length.

From our definition of a bending moment it will be seen that the bending moment is 0 at the free end and a maximum at the wall; and the tendency being to bend the beam concave downward all the bending moments are negative.

In the construction of the bending-moment diagram of Fig. 40 the following data were used

Linear scale, 0.2 inch = 1 foot; bending-moment scale, 1 inch in depth = 800 pounds-feet. $L = 10$ feet = 2 inches to scale; $r_1 = 7$ feet = 1.4 inches to scale; $r_2 = 4$ feet = 0.8 inch to scale; $W = 32$ pounds; $W_1 = 44$ pounds; $w = 48$ pounds per foot run.

If the beam were loaded only with W at the extremity the bending moment at the wall would be

$-WL = -32 \times 10 = -320$ lbs.-ft. $= -\frac{320}{800} = -0.4$ inch to scale,

which would be the length of the ordinate *no* at the wall representing the bending moment due to W, and *mno* would be the complete bending-moment diagram.

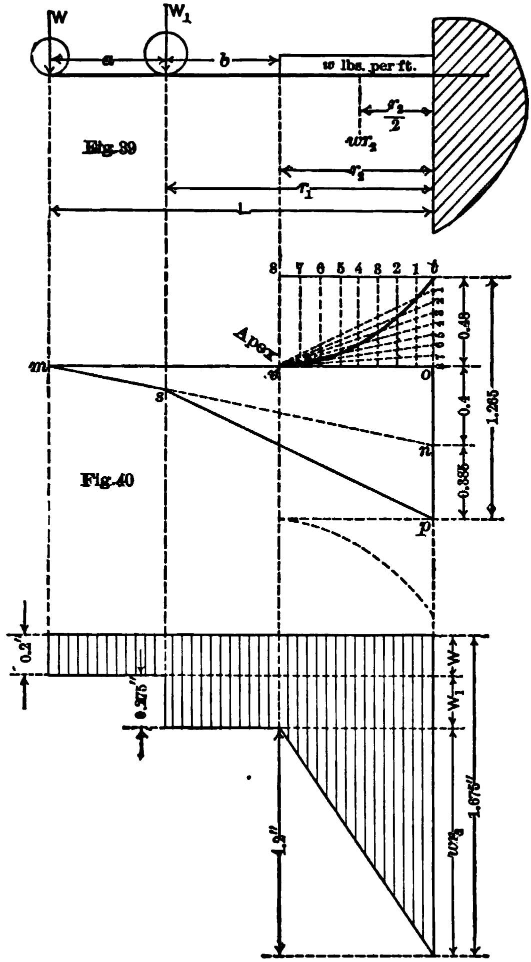


Fig. 41.

The addition of the concentrated load W_1 would occasion an additional bending moment at the wall of

$$-W_1r_1 = -44 \times 7 = -308 = -\tfrac{308}{800} = -0.385 \text{ inch to scale,}$$

which would be the length of the ordinate *np* at the wall representing the bending moment due to W_1. Drawing the line *sp* we would then have *mspo* as the complete bending-moment diagram due to W and W_1.

The addition of the uniform load occasions a bending moment at the wall of

$$-wr_2 \times \frac{r_2}{2} = -48 \times 4 \times 2 = -384 \text{ pounds-feet}$$
$$= -\tfrac{384}{800} = -0.48 \text{ inch to scale,}$$

which would be the length of the ordinate *ot* at the wall representing the bending moment due to the uniform load, taking *mo* as a base line. The locus of the extremities of the ordinates representing the bending moments at all other sections of the uniformly loaded part would be the parabola having its apex at *v* and passing through *t*.

The complete bending-moment diagram for all the loads is therefore *msptv*. This can be checked by finding at once the bending moment at the wall due to all the loads, thus:

$$M = -WL - W_1r_1 - wr_2 \times \frac{r_2}{2} = -32 \times 10 - 44 \times 7 - 48 \times 4 \times 2$$
$$= 1012 \text{ pounds-feet} = \tfrac{1012}{800} = 1.265 \text{ inches to scale.}$$

The part of the bending-moment diagram due to the uniform load being negative properly belongs below the base line *mo*, as shown by the dotted lines, but it has been placed above *mo* in order to obtain a solid diagram from which the bending moment at any section under the uniform load may be obtained by one measurement.

In the construction of the shear diagram of Fig. 41 the load scale was taken as 1 inch = 160 pounds.

If W were the only load on the beam the shear throughout the beam would be $-W = -32$ pounds $= -\frac{32}{160} = 0.2$ inch to scale. Between the point of application of W_1 and the wall the shear is increased by the amount $-W_1 = -44$ pounds $= -\frac{44}{160} = -0.275$

inch to scale. The addition of the uniform load increases the shear uniformly from zero at its commencement to $-wr_2$ pounds $= -48 \times 4 = -192$ pounds $= -\frac{192}{160} = -1.2$ inches to scale at the wall.

This can be checked by finding at once the shear at the wall due to all the loads, thus:

$$\text{Shear at wall} = -W - W_1 - wr_2 = -32 - 44 - 48 \times 4 = -268 \text{ lbs.}$$
$$= -\tfrac{268}{160} = -1.675 \text{ inches to scale.}$$

22. Beam with Overhanging Ends and Uniformly Loaded. — The beam of Fig. 42 is uniformly loaded with w pounds per unit of length and overhangs the supports a distance a at each end. Consider, quite independently of each other, the two overhangs as cantilevers and the part between the supports as a simple beam.

Since the overhangs tend to bend concave downward the bending moments due to them are negative. The bending moment at each support due to the overhanging cantilevers is $-wa \times \frac{a}{2} = -\frac{wa^2}{2}$ and, as we have seen in Art. 21, the parabolic curves joining the extremities of the beam with the extremities of the ordinates representing the bending moments at the supports form the bending-moment diagrams of the overhangs, Fig. 43. Disregarding the load on the central span of the beam, the reactions at the supports are each wa. Then the bending moment for any section between the supports and distant x from the left support is

$$M = wax - wa\left(\frac{a}{2} + x\right) = -\frac{wa^2}{2},$$

showing the bending moment due to the overhanging loads to be constant between the supports and negative in character. Therefore the negative part below the base line mn of Fig. 43 represents the bending moments of the whole beam due to the overhangs.

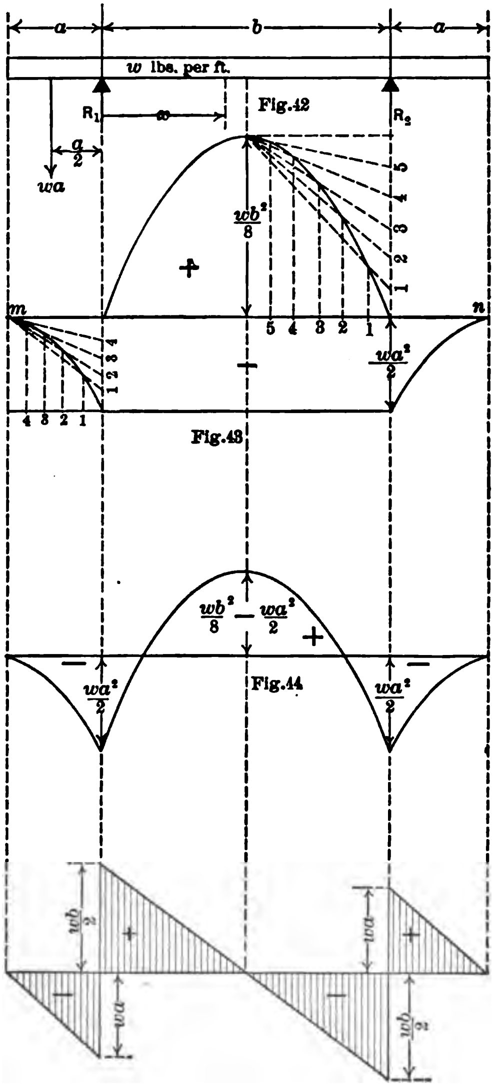

a
b
a
w lbs. per ft.
R1
x
Fig.42
R2
a/2
wa
wb²/8
+
5 4 3 2 1
m
n
1 2 3 4
4 3 2 1
wa²/2
Fig.43
wb²/8 − wa²/2
+
−
Fig.44
wa²/2
wa²/2
wb/2
+
wa
−
wb/2

Fig. 45.

The uniformly loaded central span of length b occasions a reaction at each support of $\frac{wb}{2}$, and as the tendency is to bend the beam concave upward the bending moments are positive. The bending moment at any section between the supports and distant x from the left support is

$$M = \frac{wbx}{2} - \frac{wx^2}{2} = \frac{wx}{2}(b - x),$$

which, as we have seen, is the equation of a parabola having its axis vertical over the middle of the beam. When $x = \frac{b}{2}$ we have for the maximum bending moment

$$M_{\max} = \frac{wb^2}{8}.$$

The parabola above the base line of Fig. 43 is the bending-moment diagram of the part of the beam between the supports due alone to its load, and is positive. By superposition the resultant bending-moment diagram of the whole beam, Fig. 44, is obtained.

Considering the beam in its entirety, the shear of the overhang at the left end increases uniformly from zero at the extremity to $-wa$ at the left support. The reactions due to the load are each $wa + \frac{wb}{2}$, so that at the instant of passing the left support the shear becomes

$$-wa + wa + \frac{wb}{2} = \frac{wb}{2}.$$

At the middle the shear becomes

$$-wa + wa + \frac{wb}{2} - \frac{wb}{2} = 0.$$

Immediately to the left of the right support the shear is

$$-wa + wa + \frac{wb}{2} - wb = -\frac{wb}{2}.$$

Just passing the right support the shear is

$$-wa + wa + \frac{wb}{2} - wb + wa + \frac{wb}{2} = wa.$$

At the extreme right end the shear is

$$-wa + wa + \frac{wb}{2} - wb + wa + \frac{wb}{2} - wa = 0.$$

These values of the shear enable the shear diagram of Fig. 45 to be constructed.

23. Simple Beam Loaded Uniformly over a Part of its Length Adjoining one Support. — The beam of Fig. 46 is supported at the ends and loaded with w pounds per unit of length for a distance r from the left support.

To find the support reactions we take moments about the supports, thus:

$$LR_1 = wr\left(L - \frac{r}{2}\right), \quad \text{whence} \quad R_1 = wr\left(1 - \frac{r}{2L}\right);$$

$$LR_2 = \frac{wr^2}{2}, \quad \text{whence} \quad R_2 = \frac{wr^2}{2L}.$$

The general expression for the bending moment within the limits of the loaded part of the beam is

$$M = R_1x - \frac{wx^2}{2}.$$

For $x = r$ we get

$$M = R_1r - \frac{wr^2}{2} = \frac{wr^2}{2}\left(1 - \frac{r}{L}\right)$$

as the bending moment at the section where the load ceases.

On the base line ab, Fig. 47, erect the ordinate cd under the section of the beam at the right extremity of the loaded part, and make it equal in length to $\frac{wr^2}{2}\left(1 - \frac{r}{L}\right)$ to a chosen bending-moment scale.

The general expression for the bending moment at any section between the loaded part and the right support is

$$M = R_1x_1 - wr\left(x_1 - \frac{r}{2}\right) = \frac{wr^2}{2}\left(1 - \frac{x_1}{L}\right),$$

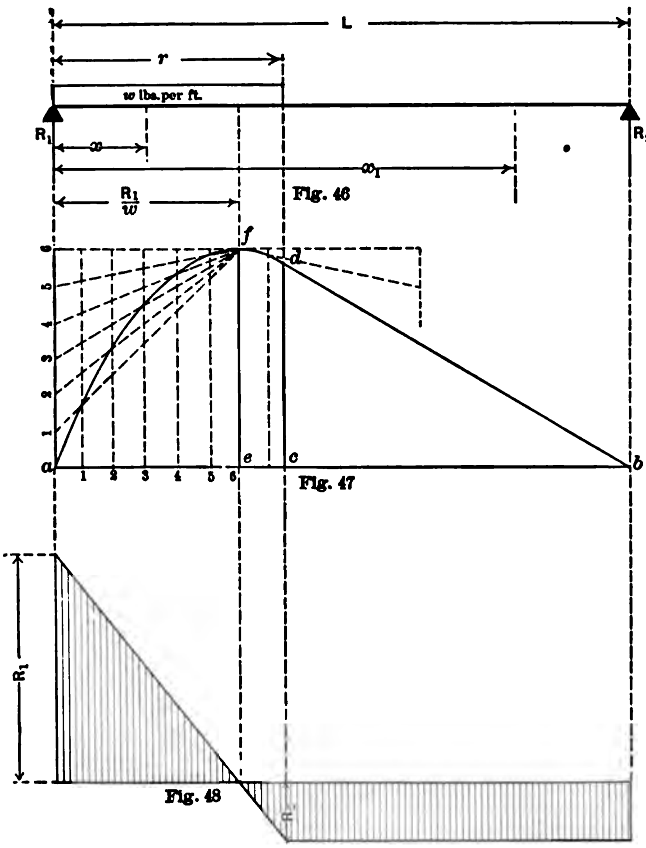


Fig. 46

Fig. 47

Fig. 48

the equation of a straight line which gives a zero value for M when $x_1 = L$. The triangle bcd, therefore, is the diagram of the bending moments for the unloaded part of the beam.

The maximum bending moment will evidently occur at some section within the limits of the load. To locate this section we proceed thus:

Within the limits of the load

$$M = R_1x - \frac{wx^2}{2},$$

$$\frac{dM}{dx} = R_1 - wx,$$

$$R_1 - wx = 0, \text{ whence } x = \frac{R_1}{w}.$$

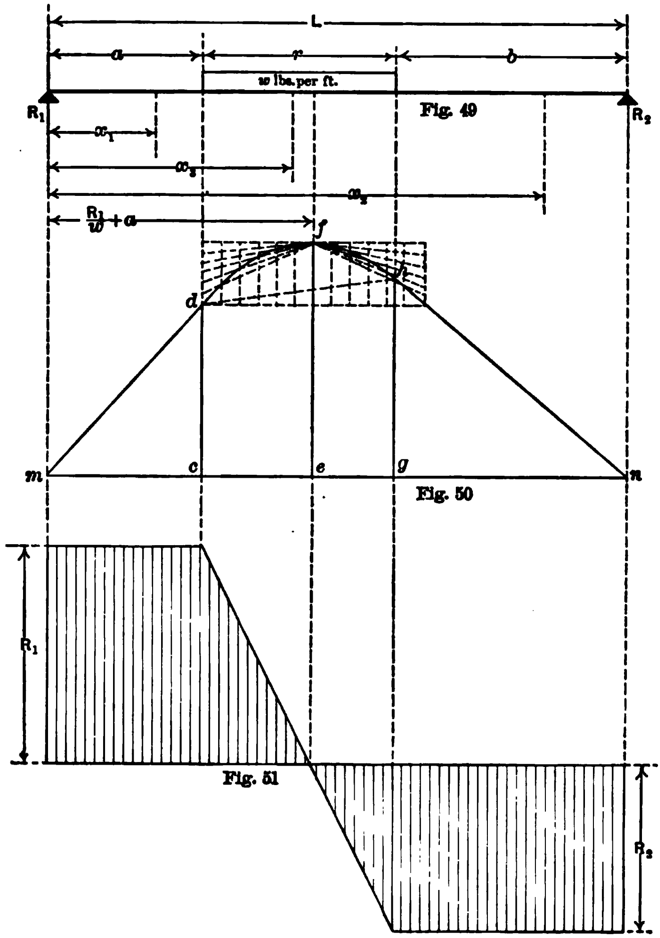
The maximum bending moment therefore occurs at the section distant $\frac{R_1}{w}$ from the left support. Substituting this value of x in the general expression for the bending moment within the limits of the load we get

$$M_{\max} = \frac{R_1^2}{w} - \frac{wR_1^2}{2\,w^2} = \frac{R_1^2}{2\,w}.$$

Under the section distant $\frac{R_1}{w}$ from the left support erect the ordinate ef and make it equal in length to $\frac{R_1^2}{2\,w}$ to the chosen scale. This ordinate is the axis of the parabola representing the bending moments due to the uniform load, and since this parabola must pass through the points a and d it is readily constructed as shown, and the bending-moment diagram of the beam completed.

The general expression for the shear over the loaded part of the beam is $R_1 - wx$, which, for $x = 0$, gives R_1 as the shear at the left support. For $x = r$ the shear becomes $R_1 - wr = wr\left(1 - \frac{r}{2\,L}\right) - wr = -\frac{wr^2}{2\,L} = -R_2$ and remains unchanged over the unloaded part of the beam. From the shear values just found the shear diagram of Fig. 48 is constructed, and it will be

observed that the maximum bending moment occurs at the section where the shear is zero.



24. Simple Beam Loaded Uniformly over a Part of its Length not Adjoining Either Support. — The beam of Fig. 49 is supported at the ends and loaded with w pounds per foot for a

distance r commencing at a distance a from the left support and terminating at a distance b from the right support.

To find the support reactions we take moments about the supports, thus:

$$LR_1 = wr\left(b + \frac{r}{2}\right), \quad \text{whence} \quad R_1 = \frac{wr}{L}\left(b + \frac{r}{2}\right);$$

$$LR_2 = wr\left(a + \frac{r}{2}\right), \quad \text{whence} \quad R_2 = \frac{wr}{L}\left(a + \frac{r}{2}\right).$$

The general expression for the bending moment between the left support and the section where the loading begins is

$$M = R_1x_1 = \frac{wrx_1}{L}\left(b + \frac{r}{2}\right),$$

the equation of a straight line. For $x_1 = 0$, $M = 0$; for $x_1 = a$, $M = \frac{wra}{L}\left(b + \frac{r}{2}\right)$. On the base line mn, Fig. 50, and under the section distant a from the left support, erect the ordinate cd and make it equal in length to $\frac{wra}{L}\left(b + \frac{r}{2}\right)$ to some chosen scale for bending moments. Join m and d and the triangle mcd is then the diagram of the bending moments of the unloaded part of the beam adjoining the left support.

The general expression for the bending moment of the unloaded part of the beam adjoining the right support is, taking moments to the right,

$$M = R_2(L - x_2) = \frac{wr}{L}\left(a + \frac{r}{2}\right)(L - x_2) = wr\left(a + \frac{r}{2}\right)\left(1 - \frac{x_2}{L}\right),$$

the equation of a straight line. For $x_2 = a + r$,

$$M = wr\left(a + \frac{r}{2}\right)\left(1 - \frac{a + r}{L}\right);$$

and for $x_2 = 0$, $M = 0$.

At a distance of $a + r$ from the left support erect the ordinate gh and make it equal in length to $wr\left(a + \frac{r}{2}\right)\left(1 - \frac{a + r}{L}\right)$. It

will represent the bending moment at the section under the right extremity of the load. Join h and n and the triangle ngh is the diagram of the bending moments of the unloaded part of the beam adjoining the right support.

The maximum bending moment evidently occurs at some section within the limits of the loaded part of the beam. To locate this section we proceed thus:

Within the limits of the load

$$M = R_1 x_3 - \frac{w}{2}(x_3 - a)^2,$$

$$\frac{dM}{dx_3} = R_1 - w(x_3 - a),$$

$$R_1 - w(x_3 - a) = 0, \quad \text{whence} \quad x_3 = \frac{R_1}{w} + a.$$

The maximum bending moment occurs, therefore, at the section distant $\frac{R_1}{w} + a$ from the left support. Substituting this value of x_3 in the general expression for the bending moment within the limits of the load we get

$$M_{\max} = \frac{R_1^2}{w} + R_1 a - \frac{w}{2}\left(\frac{R_1}{w} + a - a\right)^2 = R_1\left(\frac{R_1}{2w} + a\right).$$

Under the section distant $\frac{R_1}{w} + a$ from the left support erect the ordinate ef and make it equal to $R_1\left(\frac{R_1}{2w} + a\right)$ to the chosen scale. This ordinate is the axis of the parabola representing the bending moments due to the uniform load, and since this parabola must pass through d and h it is readily constructed as shown, and the bending-moment diagram of the whole beam completed.

The shear over the unloaded part of the beam adjoining the left support is R_1, and over the loaded part of the beam it is $R_1 - w(x_3 - a)$, which, for $x_3 = a + r$, gives for the

shear at the section at the right extremity of the load, $R_1 - wr$ $= \frac{wr}{L}\left(b + \frac{r}{2}\right) - wr = \frac{wr}{L}\left(b + \frac{r}{2} - L\right) = -\frac{wr}{L}\left(a + \frac{r}{2}\right) = -R_2$, and continues unchanged to the right support. From these shear values the shear diagram of Fig. 51 is constructed, and it will be observed that the maximum bending moment occurs at the section where the shear is zero.

EXAMPLE. — The beam of Fig. 52 is 32 feet long and overhangs the right support 6 feet and the left support 8 feet. In addition to a uniformly distributed load of 20 pounds per foot it carries concentrated loads of 200 pounds and 390 pounds as shown. Find the maximum positive and negative bending moments and construct the bending-moment and shear diagrams.

Solution. — The support reactions must first be found by taking moments about the supports; thus,

$$18\,R_1 + 20 \times 6 \times 3 = 20 \times 26 \times 13 + 200 \times 22 + 390 \times 6,$$

whence $R_1 = 730$ pounds.

$$18\,R_2 + 20 \times 8 \times 4 + 200 \times 4 = 20 \times 24 \times 12 + 390 \times 12,$$

whence $R_2 = 500$ pounds

In complicated problems like this it is the best practice to calculate the bending moments at various sections of the beam and use the results, when reduced to scale, as the lengths of ordinates in the construction of the bending-moment diagram. In this instance the linear scale will be taken as 0.1 inch = 1 foot, and the bending-moment scale as 1 inch = 1000 pounds-feet.

For the purpose of constructing the bending-moment diagram these bending moments are obtained by calculation

At the section under W_1

$$M = -20 \times 4 \times 2 = -160 \text{ pounds-feet.}$$

At the left support

$$M = -20 \times 8 \times 4 - 200 \times 4 = -1440 \text{ pounds-feet.}$$

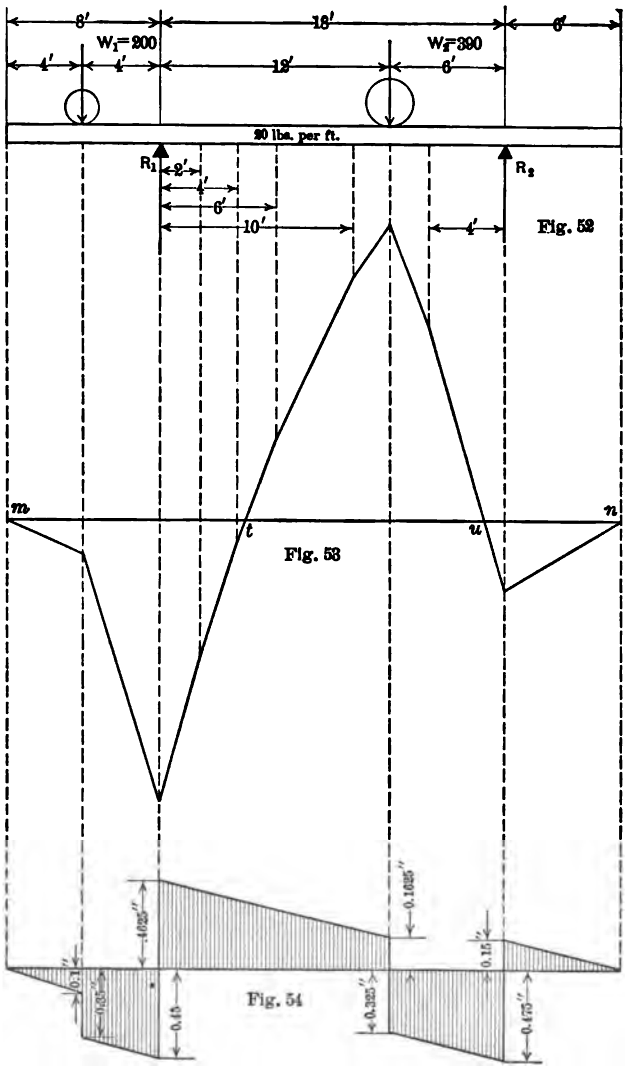


Fig. 52

Fig. 53

Fig. 54

At the section 2 feet to the right of the left support

$$M = 730 \times 2 - 20 \times 10 \times 5 - 200 \times 6 = -740 \text{ lbs.-ft.}$$

At the section 4 feet to the right of the left support

$$M = 730 \times 4 - 20 \times 12 \times 6 - 200 \times 8 = -120 \text{ lbs.-ft.}$$

At the section 6 feet to the right of the left support

$$M = 730 \times 6 - 20 \times 14 \times 7 - 200 \times 10 = 420 \text{ lbs.-ft.}$$

At the section 10 feet to the right of the left support

$$M = 730 \times 10 - 20 \times 18 \times 9 - 200 \times 14 = 1260 \text{ lbs.-ft.}$$

At the section under W_2

$$M = 730 \times 12 - 20 \times 20 \times 10 - 200 \times 16 = 1560 \text{ lb.-ft.}$$

At the section 4 feet to the left of the right support

$$M = 730 \times 14 - 20 \times 22 \times 11 - 200 \times 18 - 390 \times 2 = 1000 \text{ pounds-feet.}$$

At the right support

$$M = 730 \times 18 - 20 \times 26 \times 13 - 200 \times 22 - 390 \times 6 = -360 \text{ pounds-feet.}$$

At the right end

$$M = 730 \times 24 - 20 \times 32 \times 16 - 200 \times 28 - 390 \times 12 + 500 \times 6 = 0.$$

Reducing these bending-moment results to scale by dividing each by 1000, and using the results as the lengths of ordinates with respect to *mn* as an axis, the bending-moment diagram of Fig. 53 was constructed.

The maximum positive bending moment is seen to be at the section under W_2, the ordinate measuring 1.56 inches, which, reduced to scale, represents 1560 pounds-feet. The maximum negative bending moment is under the left support and measures 1440 pounds-feet. The bending moment which is numerically greatest, whether positive or negative, is the one to be considered in the design of the beam.

The shear diagram of Fig. 54 may be obtained by taking the algebraic sum of the forces to the left of different sections, thus:

Commencing at the left end, the shear increases uniformly and negatively at the rate of 20 pounds per foot from the value zero until, at the section immediately to the left of W_1, it becomes

$$-20 \times 4 = -80 \text{ pounds,}$$

which, to a scale of 800 pounds to the inch, is represented in the diagram of Fig. 54 by -0.1 inch.

Just passing W_1 the shear becomes

$$-80 - 200 = -280 \text{ pounds,}$$

which, to scale, is represented by -0.35 inch.

At the section just to the left of the left support the shear becomes

$$-20 \times 8 - 200 = -360 \text{ pounds,}$$

which, to scale, is represented by -0.45 inch.

Just passing the left support the shear becomes

$$-20 \times 8 - 200 + 730 = 370 \text{ pounds,}$$

which, to scale, is represented by 0.4625 inch; and so on to the right end of the beam.

The shears at the different sections may be obtained also by taking the first derivatives of the bending moments at the sections, thus:

Reckoning from the left end of the beam, the expression for the bending moment for any section up to W_1 is $-\dfrac{20x^2}{2}$.

Then $$\frac{dM}{dx} = -20x = -20 \times 4 = -80 \text{ pounds}$$

for the shear just to the left of W_1.

For any section between W_1 and the left support

$$M = -\frac{20x^2}{2} - 200(x - 4).$$

Then $\frac{dM}{dx} = - 20x - 200 = - 20 \times 4 - 200 = - 280$ pounds

for the shear at the section just to the right of W_1; and

$$\frac{dM}{dx} = - 20x - 200 = - 20 \times 8 - 200 = - 360 \text{ pounds}$$

for the shear at the section just to the left of the left support.

For any section between the left support and W_2

$$M = R_1 (x - 8) - W_1 (x - 4) - \frac{20x^2}{2}$$

Then $\frac{dM}{dx} = R_1 - W_1 - 20x = 730 - 200 - 20 \times 8 = 370$ lbs.

for the shear at the section just to the right of the left support; and so on to the right end of the beam.

The completion of the shear diagram by either method is left as a study for the student.

25. Moving Loads. — A load moving over a structure, such as a train moving over a bridge, is a *live* load, sometimes known as a *rolling* load. A study of the bending moment and shear effects of the different conditions of loads moving over structures is beyond the scope of this work, and only the simple case of a single concentrated load moving over a beam will here be given. For an extended study of moving loads recourse must be had to a more advanced treatise on the *Mechanics of Materials.*

26. Simple Beam with a Concentrated Moving Load. — Neglecting the weight of the beam of Fig. 55, suppose the load W to move from left to right.

By moments about the right support we find $R_1 = \frac{W(L - x)}{L}$.

While the load is between the left support and some section a distant r from the support, the bending moment is

$$M = R_1 r - W(r - x) = \frac{Wr(L - x)}{L} - W(r - x) = Wx\left(1 - \frac{r}{L}\right).$$

This result must be positive, since r is less than L, and will

increase in value as x increases, until the load reaches a, where r becomes x, and we shall have for the bending moment at the load,

$$M = Wx - \frac{Wx^2}{L}. \tag{3}$$

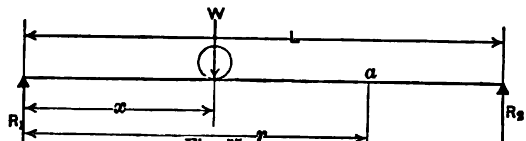


Fig. 55

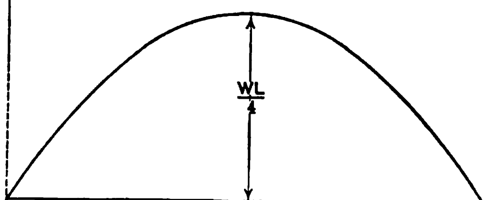


Fig. 56

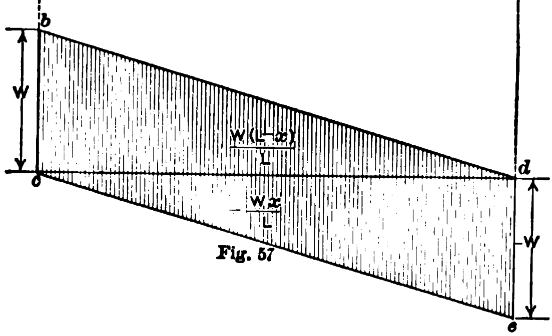


Fig. 57

When the load passes section a and moves toward the right support, the bending moment at a becomes R_1r and decreases in value, since R_1 becomes less as W approaches the right support. When W reaches the right support, R_1 becomes zero and the bending moment at a becomes zero. The bending

moment at a is greatest when the load is at a, and therefore the bending moment at any section is a maximum when the load is at the section.

To find the dangerous section, the value of x in equation (3) which makes M a maximum must be found. Thus,

$$\frac{dM}{dx} = W - \frac{2Wx}{L} = 0; \quad \text{whence,} \quad x = \frac{L}{2}.$$

The maximum bending moment at the load, and therefore at the dangerous section, is then at the middle of the beam.

Since the maximum bending moment at any section occurs when the load is at the section, the diagram of maximum bending moments, Fig. 56, is constructed from equation (3), which is that of a parabola, the maximum ordinate $\frac{WL}{4}$ being at the middle, where $x = \frac{L}{2}$.

The shear diagram is shown in Fig. 57. At the instant the load starts to move to the right from the left support, $R_1 = W$ and $R_2 = 0$. As the movement continues the shear V to the left of the load is

$$V = R_1 = \frac{W(L-x)}{L}, \tag{4}$$

a straight-line equation. At the left support $x = 0$ and $V = W$; at the right support $x = L$ and $V = 0$. The line of equation (4) is then bd, and the diagram for the shear to the left of the load takes the triangular form bcd.

To the right of the load during the movement the shear is

$$V = R_1 - W = -R_2 = -\frac{Wx}{L}, \tag{5}$$

a straight-line equation. At the left support $x = 0$ and $V = 0$; at the right support $x = L$ and $V = -W$. The line of equation (5) is then ce, and the diagram for the shear to the right of the load takes the triangular form ced.

27. Bending-moment Influence Line. — An ordinate of the bending-moment diagram of Fig. 56 represents the *greatest* bending moment produced by the rolling load at the section corresponding to the ordinate, but it is obvious that as the load moves over the beam the bending moment at the section changes continuously. The graphic representation of the variation of the bending moment due to a rolling load at a given section of a beam or structure is called the *influence line* of the bending moment at the section.

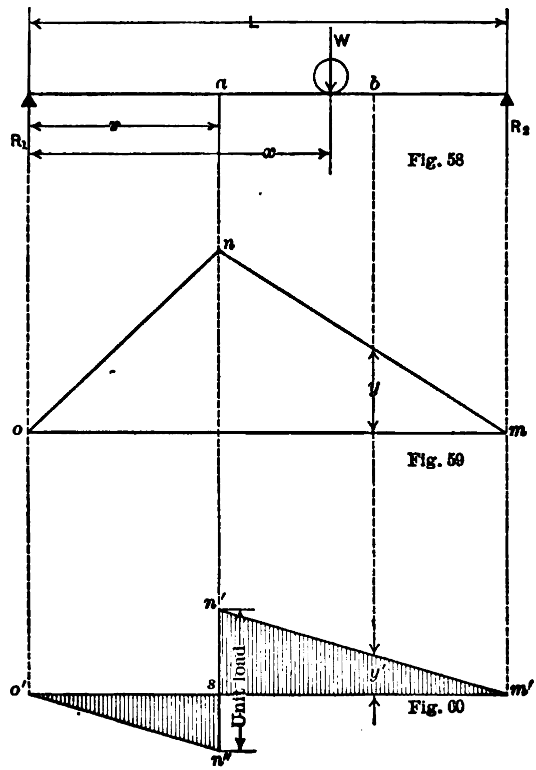


Fig. 58

Fig. 59

Fig. 60

Let x denote the distance from the left support of a concentrated load W moving over the beam of Fig. 58, and let

r denote the distance of a given section a from the left support.

While W is at the right of a we shall have

$$M_a = R_1 r = \frac{Wr(L-x)}{L}.$$

If W be taken as the unit load of one pound or of one ton we shall have

$$M_a = \frac{r(L-x)}{L}, \tag{6}$$

the equation of the straight line mn, Fig. 59, the locus of all values of M_a between the limits of $x = L$ and $x = r$.

When the load moves to the left of a we have

$$M_a = R_1 r - W(r-x) = \frac{Wr(L-x)}{L} - W(r-x) = \frac{Wx(L-r)}{L}.$$

For the unit load we shall then have

$$M_a = \frac{x(L-r)}{L}. \tag{7}$$

Equation (7) is that of the straight line on, which is the locus of the values of M_a between the limits $x = r$ and $x = 0$.

For $x = r$ in equations (6) and (7) we have the same value, $M_a = \frac{r(L-r)}{L}$; hence the two lines intersect at n, and mno is the influence line of the bending moment at section a. The ordinate to the influence line under any section measures the bending moment at a due to the unit load when moving over the section, and the whole moment is found by multiplying the ordinate by W. Thus, the bending moment at section a when the load passes over section b is Wy.

When the load comes on the beam from the right, the bending moment at a increases uniformly from zero to the maximum at n, and then decreases uniformly to the value zero at the left support.

28. Shear Influence Line. — The influence line for the shear at section a of Fig. 58 may be expressed graphically.

While the moving load W is to the right of a the shear V at a is

$$V_a = R_1 = \frac{W(L-x)}{L}\text{; and for the unit load } V_a = \frac{L-x}{L}. \quad (8)$$

Equation (8) is that of the straight line $m'n'$, Fig. 60, the locus of the shear between the limits $x = L$ and $x = r$.

When the load moves to the left of a the shear is

$$V_a = R_1 - W = -R_2 = -\frac{Wx}{L}\text{; and for the unit load, } V_a = -\frac{x}{L}. \quad (9)$$

Equation (9) is that of the straight line which is the locus of V_a between the limits $x = r$ and $x = 0$.

The slopes of equations (8) and (9) being the same, the lines are parallel, and $o'n''$ parallel to $n'm'$ is the straight line of equation (9), and the broken line $m'n'n''o'$ is the influence line of the shear at the section a while the load moves over the beam.

When the unit load comes on the beam from the right the shear at a increases uniformly from zero to the value sn' at the section immediately to the right of a. As the load passes a the shear suddenly drops by the amount of the unit load and becomes negative, and then uniformly decreases numerically until it becomes zero at the left support.

The ordinate y' measures the shear at section a when the unit load passes over section b, and the shear at a due to the whole load is Wy'.

PROBLEMS

1. A uniform beam 25 feet in length, whose weight is disregarded, is supported at the ends and has a concentrated load of 400 pounds at 9 feet from the left support, and one of 500 pounds at 18 feet from the left support. Construct the bending-moment and shear diagrams.

2. Construct the bending-moment and shear diagrams of the beam of problem 1, taking into consideration the weight of the beam, which is 500 pounds.

3. A beam 12 feet long is supported at the ends and loaded with a weight of 3 tons at a point 2 feet from one end. Find the bending moment and shear at the middle of the beam. *Ans.* 3 tons-ft.; 0.5 ton.

4. A cantilever projects 10 feet from a wall and carries a uniform load of 60 pounds per foot; it also supports three concentrated loads of 100 300, and 500 pounds at distances from the wall of 2, 5, and 9 feet respectively. Construct the bending-moment and shear diagrams.

5. A beam overhangs both supports equally, carries a uniform load of 80 pounds per foot, and has a load of 1000 pounds in the middle. The length of the beam is 15 feet, and the distance between the supports 8 feet. Construct the bending-moment and shear diagrams.

6. A beam with equal overhanging ends is loaded with three equal concentrated loads, one at each end and one at the middle. Find the distance between the supports, in terms of the total length L, when the bending moment at the middle is equal to the bending moment over the supports. At what sections is the bending moment zero?

Ans. $\frac{4L}{5}$; $\frac{L}{5}$ from each support.

7. A beam 32 feet long and supported at the ends is loaded uniformly for a distance of 20 feet from the left support with 0.8 ton per foot. Locate the dangerous section, and find by calculation the maximum bending moment and the bending moments at the middle of the load, at the end of the load, and at the section 5 feet from the right support. Construct the bending-moment and shear diagrams.

Ans. 13.75 ft. from left support; 75.625 tons-ft.; 70 tons-ft.· 60 tons-ft.; 25 tons-ft.

8. A beam 30 feet long and supported at the ends is loaded uniformly with 0.4 ton per foot for a distance of 10 feet, the load commencing at a distance of 8 feet from the left support and terminating at the section 12 feet from the right support. Locate the dangerous section, and find by calculation the maximum bending moment and the bending moments at the extremities of the load. Construct the bending-moment and shear diagrams.

Ans. 13.667 ft. from the left support; 24.56 tons-ft.; 20.8 tons-ft.; 18.14 tons-ft.

CHAPTER III

THE THEORY OF BEAMS. BEAM DESIGN

29. Stress and Strain. — In the discussion of the theory of beams it is important to distinguish between stress and strain; that is, it must be remembered that the change of form which a load produces in a body is called the *strain*, or deformation, of the body due to the load; and that the internal, or molecular, resistance which the material of a body interposes to resist deformation is called the *stress*.

30. Coefficient of Elasticity and Hooke's Law. — Within the elastic limit of all materials the stresses are proportional to the strains, but since the same intensity of stress does not produce the same strain in different materials we must have some definite means of expressing the amount of strain produced in a body by a given stress. The means employed is to assume the body to be perfectly elastic and then state the intensity of stress necessary to strain the body by an amount equal to its own length. This stress is known as the *Modulus of Elasticity*, or the *Coefficient of Elasticity*, and is denoted by E. The values of E for different materials have been determined and are practically the same for tension and compression. The mean values for E in pounds per square inch for the materials most commonly used in engineering are as follows: Timber, 1,500,000; cast iron, 15,000,000; wrought iron, 25,000,000; steel, 30,000,000.

There are no materials of construction that are perfectly elastic, and but few that will stretch 0.001 of their lengths and remain elastic; but within the elastic limit all bodies are assumed to be perfectly elastic, and if that limit be not exceeded the

coefficient of elasticity is the ratio of the unit stress to the unit strain. This is in accordance with the discovery made in 1678 by Robert Hooke and is known as *Hooke's law.* This law expresses the fact that the ratio of the unit stress to the unit deformation is a constant quantity, known as the ***Modulus of Elasticity***. Expressed as a formula, we have

$$\frac{\text{Unit stress}}{\text{Unit deformation}} = E.$$

Cast iron, cement, and concrete are among the few materials that do not conform to Hooke's law, their coefficients of elasticity varying under different stresses; but it is customary in all investigations to assume the truth of the law, modifying results by factors of safety.

In the case of shear stress the modulus is known as that of *transverse elasticity* or *coefficient of rigidity*, commonly denoted by G. The mean values for G in pounds per square inch for the common engineering materials are: Timber, 150,000; cast iron, 5,500,000; wrought iron, 10,500,000; steel, 12,500,000.

31. The Determination of E. — If y denotes the strain produced in a body by a stress S, and Y the strain produced by the stress necessary to stretch the body to double its length, then, since the stresses are proportional to the strains, we shall have from the stress-strain diagram, Fig. 61,

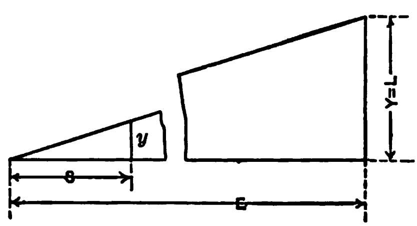


Fig. 61.

$$\frac{y}{Y} = \frac{S}{E}, \quad \text{whence} \quad E = \frac{SY}{y} = \frac{SL}{y},$$

since the strain, or deformation, Y, is equal to L.

32. Neutral Axis. — When a beam, Fig. 62, is subjected to a bending moment its axis is deflected from its original straight position into one of curvature, known as the elastic curve. The fibers in the horizontal layers on the convex side are in tension, and those on the concave side in compression. There must therefore be a layer, or surface, of fibers that is neither in tension nor in compression, called the *neutral surface,* and the line

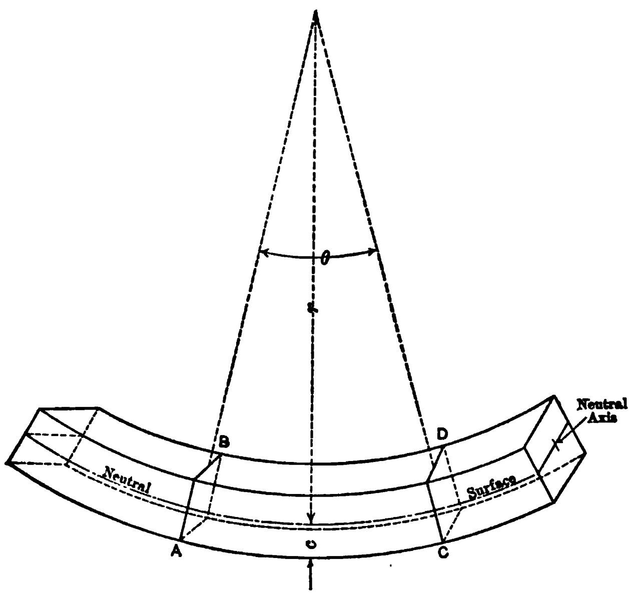


Fig. 62.

in which this surface intersects the plane of a section of the beam is the *neutral axis* of the section. The neutral axis invariably passes through the center of gravity of a section whose material has its modulus of elasticity the same in tension as in compression.

Consider the portion of the beam included between the sections AB and CD to be bent to the arc of a circle, and let r be the radius of the neutral surface, and c the distance from the

neutral surface to the outermost surface. The length of the neutral surface will not change in the bending, hence $r\theta$ denotes the original length of the outermost surface expressed in radians, and $(r + c)\,\theta$ expresses the length of the outermost surface after bending, hence

$$\text{Deformation} = (r + c)\,\theta - r\theta = c\theta.$$

But we have seen that

$$\frac{\text{Deformation}}{\text{Original length}} = \frac{S}{E},$$

hence

$$\frac{c\theta}{r\theta} = \frac{c}{r} = \frac{S}{E}. \tag{10}$$

That is, $\frac{E}{r} = \frac{S}{c}$ = unit stress at the distance unity from the neutral surface, S being the unit stress, tension or compression, of the fibers at the outermost surface, and c the distance from the neutral surface to the outermost surface.

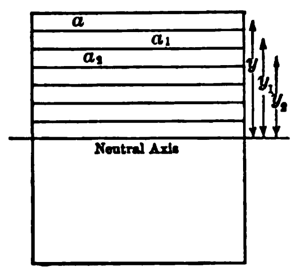


Fig. 63.

33. Resisting Moment. — Suppose a beam section, Fig. 63, to be made up of a great number of layers a, a_1, a_2, etc., distant y, y_1, y_2, etc. from the neutral axis. The unit stresses at distances y, y_1, y_2, etc., are $\frac{S}{c} \times y$, $\frac{S}{c} \times y_1$, $\frac{S}{c} \times y_2$, and the total stresses on the elemental areas are $\frac{S}{c} \times ay$, $\frac{S}{c} \times a_1y_1$, $\frac{S}{c} \times a_2y_2$, etc. The moments of these stresses are $\frac{S}{c} \times ay^2$, $\frac{S}{c} \times a_1y_1^2$, $\frac{S}{c} \times a_2y_2^2$, etc. The sum of all these moments will be the resisting moment of the section, hence

$$\text{Resisting moment} = \frac{S}{c}\,(ay^2 + a_1y_1^2 + a_2y_2^2 + \text{etc.}) = \frac{SI}{c},$$

in which I is the moment of inertia of the section about the neutral axis. The bending moment must be equal to the resisting moment, since for equilibrium the internal stresses must be equal to the external forces; therefore

$$\text{Bending moment} = M = \frac{SI}{c}. \tag{11}$$

The equation $M = \frac{SI}{c}$ is the fundamental equation for beam investigations, and since I is expressed in bi-quadratic inches, M must be expressed in pounds-inches. The length of a required beam and the load to which it is to be subjected being known, we can readily find the maximum bending moment, and then by assuming the allowable working stress, S, per unit of area at the outermost fiber, we shall have $\frac{M}{S} = \frac{I}{c}$. The numerical value thus obtained for $\frac{I}{c}$ must be satisfied by the dimensions of the cross section of the beam. The smaller value of S, whether for tension or compression, should be taken.

34. Section Modulus. — The factor $\frac{I}{c}$ contains the dimensions of the section and is a comparative measure of its strength. It is called the *modulus* of the section and is denoted by Z. From equation (11) we have

$$\frac{M}{S} = \frac{I}{c} = Z, \quad \text{whence} \quad S = \frac{M}{Z}.$$

Substituting this value of S in equation (10), we have

$$\frac{c}{r} = \frac{M}{EZ}, \quad \text{whence} \quad M = \frac{EZc}{r} = \frac{EI}{r}, \tag{12}$$

which is the equation of the elastic curve, showing the relationship between the bending moment at any section and the radius of curvature of the beam.

35. Assumptions in the Theory of Beams. — In the theory of beams there are three important assumptions made: (*a*) That the strain increases directly as the distance from the neutral axis; (*b*) that the stress varies as the strain; (*c*) that the coefficient of elasticity is the same for tension as for compression.

No one of these assumptions is perfectly true, but for nearly all materials they are so nearly true that for practical purposes they may be regarded as true, so long as the elastic limit is not exceeded.

36. Beam Design. — Beams are divided into two general classes: Those of uniform strength and those of uniform cross section.

Beams of uniform strength are so shaped that the unit stress in the surface fiber is the same at all cross sections. They are employed when it is desired to reduce to a minimum the weight of the material in a beam. In their design the section modulus must vary directly with the bending moment, since the unit fiber stress remains constant.

Beams of uniform cross section are those ordinarily used, as they are readily sawed from wood or rolled from steel. In such beams the section modulus remains constant and the fiber stress varies directly with the bending moment.

A beam is designated by its depth, and its design is a question of determining the sectional dimensions which will bear with safety a given load, the length of the beam being known.

If the dimensions of the beam are to be determined from the maximum allowable fiber stress, the bending moment at the dangerous section must be computed, after which the section modulus, $\frac{I}{c}$, may be found from the equation $M = \frac{SI}{c}$. If it is desired to use one of the standard types manufactured by the different steel companies, the beam in the manufacturers' tables

having an $\frac{I}{c}$ equal to or next greater than the one found is to be selected.

The standard I beams so extensively used in structural work are rolled in light, intermediate, and heavy weights of thirteen sizes. The different manufacturers issue handbooks containing tables of the properties of the beams they produce, and while there is an agreement in sizes as to depth, there are marked differences in the proportions of cross sections, and therefore also of weight per foot and of moments of inertia.

The upper and lower horizontal parts of an I section are called the *flanges*, and the vertical part connecting them is called the *web*. In the design of an I beam the web is assumed to resist the vertical shear, the flanges alone resisting the bending moment.

For the support of loads beyond the capacity of a single I beam, two or more of them may be bolted together side by side, in which case cast-iron separators are used, fitting neatly between the flanges and bolted to the webs so as to unite the beams forming the girder and cause them to act together in resisting the load. These separators should be placed at each end of the girder and at points where loads are concentrated. For uniform loading the spacing of the separators should not be greater than twenty times the width of the smallest beam flange in order to prevent failure by buckling at the upper flanges, which are in compression.

For the support of still greater loads a useful and economical section can be formed by the combination of two I beams having plates riveted to their top and bottom flanges, forming what is known as a *box girder*.

Should the cross section of a beam be other than those listed as standard, its dimensions may be determined by first computing the maximum bending moment and then, knowing the safe

allowable value of S, one dimension is assumed and the other computed.

EXAMPLE I. — A yellow pine beam of 18 feet span is to support concentrated loads of 1200 pounds and 1500 pounds at points 4 feet and 10 feet from the left support. Taking the ultimate tensile strength of yellow pine as 9000 pounds per square inch, determine the cross-section dimensions of the beam.

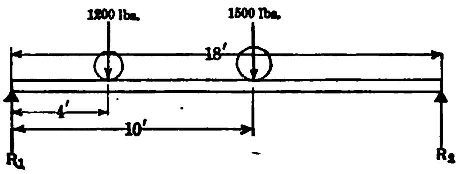


Fig. 64.

Solution. — From Fig. 64 we have

$18\,R_1 = 1200 \times 14 + 1500 \times 8,$ whence $R_1 = 1600$ pounds; also

$18\,R_2 = 1500 \times 10 + 1200 \times 4,$ whence $R_2 = 1100$ pounds.

The construction of the shear diagram shows the shear to pass through zero at the section under the 1500-pound load, thus locating the dangerous section. Then

$M_{max} = 1600 \times 10 - 1200 \times 6 = 8800$ lbs.-ft. $= 105{,}600$ lbs.-ins.

Using a factor of safety of 10, we shall have

$$S = \frac{9000}{10} = 900 \text{ pounds per square inch}$$

as the fiber stress at the outermost surface. Assuming a breadth of 5 inches, and remembering that $c = \frac{d}{2}$, we have

$$\frac{M}{S} = \frac{105{,}600}{900} = \frac{I}{c} = \frac{bd^3}{12\,c} = \frac{5\,d^2}{6}, \quad \text{whence} \quad d = 12 \text{ nearly},$$

so that a 5-inch by 12-inch beam would appear safe.

We shall, however, investigate the influence of the weight of the beam itself.

Yellow pine weighs 40 pounds per cubic foot, hence

$$\text{Weight of beam} = \frac{5 \times 12 \times 18 \times 12 \times 40}{1728} = 300 \text{ pounds}$$
$$= 16.66 \text{ pounds per foot.}$$

Then R_1 becomes $1600 + 150 = 1750$ pounds, and we shall have

$$M_{max} = 1750 \times 10 - 1200 \times 6 - 16.66 \times 10 \times 5 = 9466.66 \text{ lbs.-ft.}$$
$$= 113{,}600 \text{ pounds-inches.}$$

Then $$S = \frac{Mc}{I} = \frac{113{,}600 \times 6 \times 12}{5 \times 1728} = 947 \text{ lbs. per square inch,}$$

which is greater than the assumed safe stress of 900 pounds, so a 5.5-inch by 12-inch beam will be tried. Then

$$\text{Weight of beam} = \frac{5.5 \times 12 \times 18 \times 12 \times 40}{1728} = 330 \text{ pounds}$$

$= 18.33$ pounds per foot, and $R_1 = 1765$ pounds. Then

$$M_{max} = 1765 \times 10 - 1200 \times 6 - 18.33 \times 10 \times 5 = 9533.5 \text{ lbs.-ft.}$$
$$= 114{,}402 \text{ pounds-inches.}$$ Then

$$S = \frac{Mc}{I} = \frac{114{,}402 \times 6 \times 12}{5.5 \times 1728} = 867 \text{ pounds per square inch,}$$

a result which shows the 5.5-inch by 12-inch beam to be safe.

Example II. — What Cambria I beam should be selected for a floor bearing a load of 180 pounds per square foot, the beams to have a span of 25 feet, to be spaced 8 feet apart, and to have a maximum unit stress of 16,000 pounds per square inch?

Solution. —

Load on each beam $= 25 \times 8 \times 180 = 36{,}000$ pounds $= \frac{36{,}000}{25}$ $= 1440$ pounds per foot run. $R_1 = 18{,}000$ pounds, and

$$M_{max} = 18{,}000 \times 12.5 \times 12 - 1440 \times 12.5 \times 6.25 \times 12$$
$$= 1{,}350{,}000 \text{ pounds-inches.}$$

Then $$\frac{I}{c} = \frac{M}{S} = \frac{1{,}350{,}000}{16{,}000} = 84.37.$$

A reference to the Cambria handbook shows that the 15-inch special I beam of 65 pounds per foot of the Cambria Steel Company should be selected, its tabulated modulus being 84.8.

EXAMPLE III. — What is the proper size of I beam to carry a load of 35,000 pounds concentrated at the middle of a span of 25 feet, the fiber stress not to exceed 16,000 pounds per square inch?

Solution. —

$$M_{\max} = 17{,}500 \times 12.5 \times 12 = 2{,}625{,}000 \text{ pounds-inches.}$$

$$\frac{I}{c} = \frac{M}{S} = \frac{2{,}625{,}000}{16{,}000} = 164.$$

This value of the section modulus would indicate that a 24-inch standard Cambria I beam of 80 pounds per foot should be selected, its tabulated section modulus being 173.9. Its tabulated moment of inertia is 2087.2.

We shall investigate the effect of the weight of the beam itself.

Weight of beam = 80 × 25 = 2000 pounds; therefore R_1 = 18,500 pounds.

Then $M_{\max} = 18{,}500 \times 12.5 \times 12 - 80 \times 12.5 \times 6.25 \times 12$
$= 2{,}700{,}000$ pounds-inches.

$$S = \frac{Mc}{I} = \frac{2{,}700{,}000 \times 12}{2087.2} = 15{,}500 \text{ pounds,}$$

which is 500 pounds less per square inch than the safe allowable stress at the outermost fiber. The 24-inch standard Cambria I beam weighing 80 pounds per foot, is, therefore, the proper selection. (The safe loads for beams given in the handbooks issued by the different steel companies include the weight of the beam.)

EXAMPLE IV. — Heavy 12-inch Cambria I beams of 20 feet span are to be used to support a floor bearing a uniform load, including the weight of the beam, of 200 pounds per square foot. The outermost fiber stress is not to exceed 16,000 pounds per square inch. Find the proper spacing of the beams.

Solution. — From the Cambria handbook we find for the beam selected,

$$\text{Section modulus} = \frac{I}{c} = 41.$$

Let x denote the distance between two consecutive beams. Then

$$\text{Load on one beam} = 20\,x \times 200 = 4000\,x \text{ pounds, and}$$

$$\text{Load per foot} = \frac{4000\,x}{20} = 200\,x \text{ pounds,}$$

$$R_1 = 2000\,x \text{ pounds.}$$

$$M_{max} = 2000\,x \times 10 \times 12 - 200\,x \times 10 \times 5 \times 12 = 120{,}000\,x \text{ lbs.-ins.}$$

Then $\frac{I}{c} = \frac{M}{S}$, or $41 = \frac{120{,}000\,x}{16{,}000}$, whence $x = 5.47$ feet.

PROBLEMS

1. A rectangular beam 9 inches deep, 3 inches wide, supports a load of 0.5 ton concentrated at the middle of an 8-foot span. Find the maximum fiber stress. *Ans.* 0.3 ton per sq. in.

2. A square timber beam of 12-inch side weighs 50 pounds per cubic foot, is 20 feet long, and supports a load of 2 long tons at the middle of its span. Calculate the fiber stress at the middle section. *Ans.* 1037 lbs. per sq. in.

3. Light 12-inch Cambria I beams of 20-feet span are to support a floor bearing a uniform load of 175 pounds per square foot. The outermost fiber stress is not to exceed 16,000 pounds per square inch. At what distance apart should the beams be placed? *Ans.* 5.5 ft.

4. What safe uniform load may be placed on a wooden cantilever 5 feet long, 3 inches wide, and 5 inches deep in order that the fiber stress shall be 900 pounds per square inch? *Ans.* 71 lbs. per ft.

5. What safe uniformly distributed load may be placed on a standard Cambria channel beam weighing 20 pounds per foot, the span being 16 feet, the web placed vertical, and the maximum fiber stress not to exceed 12,500 pounds per square inch? *Ans.* 11,150 lbs.

6. Find the factor of safety of a heavy 12-inch Cambria I beam of 15 feet clear span when sustaining a uniformly distributed load of 30 tons. *Ans.* 2.

7. Compare the strength of a steel beam of rectangular section, 5 inches by 2 inches, with that of a 9-inch I beam of the same material, length, and area of section, the moment of the inertia of the I beam being 111.8. *Ans.* I beam is three times as strong.

CHAPTER IV

THE DEFLECTION OF BEAMS

37. Deflection of Beams. — In Chapter III the question of beam design was considered from the standpoint of the maximum allowable fiber stress, but the safe load for a beam to carry may equally well be determined from its maximum allowable deflection. For the protection of plastered ceilings in buildings, the floor beams are limited in deflection to about $\frac{1}{360}$ of their length, and their design, therefore, is based on considerations of deflection rather than of strength.

The radius of curvature of any plane curve is given in the differential calculus as

$$r = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

In the case of a beam the curvature is assumed to be small, and therefore the slope of the tangent at any point is small. In other words, $\frac{dy}{dx}$ being very small, $\left(\frac{dy}{dx}\right)^2$ is infinitesimal and may be neglected, and therefore

$$r = \frac{1}{\frac{d^2y}{dx^2}}.$$

Substituting this value of r in equation (12) of Art. 34, p. 66, we have

$$M = \frac{EId^2y}{dx^2},$$

which is the equation of the elastic curve for the investigation of the deflection of beams. In any particular case the value of M must be expressed in terms of x, and then, after two integrations, the deflection, y, will be known for any value of x.

38. **Points of Inflection.** — Simple beams that are *free* at their end supports — that is, having no restraining influence to keep their supported ends horizontal — bend concave upward, but beams that are *fixed* at their end supports — that is, having their ends built in so that the built-in part is constrained to remain horizontal — are subjected to a combined curvature of bending which is concave downward near the supports and concave upward toward the middle. A beam that overhangs its supports, such as that of the Example, p. 51,

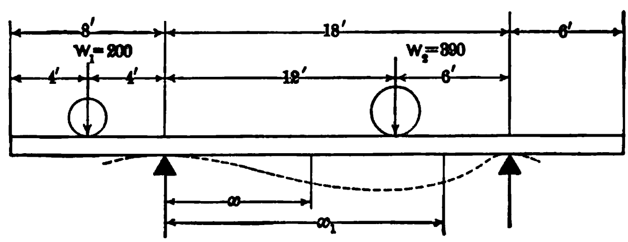


Fig. 65.

Chap. II, reproduced in Fig. 65, is also subjected to a combined curvature of bending between the supports. The overhangs exert a constraining influence to keep horizontal the parts of the beam resting on the supports and thus compel a bending of the beam concave downward near to and inside each support. This is clearly indicated by the dotted curves of Fig. 65. In these cases of double curvature in bending, the points at which the contrary curves separate and at which there is no bending whatever, are known as *points of inflection* or *points of contrary flexure*. They can be found by placing the general expression

for the bending moment within their limits equal to zero. Thus, in Fig. 65, it is evident from inspection that there will be a point of inflection between the left support and W_2, and one between the right support and W_2. The expression for the bending moment at any section within the limits of the left support and W_2 is

$$M = R_1 x - W_1 (x + 4) - \frac{w}{2} (x + 8)^2.$$

Placing this expression equal to zero, and substituting 730 for R_1, 200 for W_1, and 20 for w (see p. 51), the resulting quadratic gives a value of 4.42 for x, which shows there is a point of inflection at 4.42 feet to the right of the left support. The expression for the bending moment at any section within the limits of W_2 and the right support is

$$M = R_1 x_1 - W_1 (x_1 + 4) - W_2 (x_1 - 12) - \frac{w}{2} (x_1 + 8)^2.$$

Placing this expression equal to zero and substituting 390 for W_2 and 500 for R_2, the resulting quadratic gives for x_1 the value of 17.027, showing the other point of inflection to be at the section distant about 1 foot to the left of the right support. These points of inflection are indicated at t and u in the bending-moment diagram of Fig. 53, p. 52, at which points there are no bending moments.

The determination of the points of inflection is very important in a mechanical sense as furnishing the logical location for expansion joints in bridge construction.

39. **Examples in Beam Deflection.** — The expressions for the maximum deflections of beams under several different conditions of support and loading will here be derived; those for other conditions may be found by similar processes.

Example I. Simple beam with a load W concentrated at the middle, Fig. 66.

For any point between the left support and the middle of the beam, we have

$$M = \frac{Wx}{2}.$$

Then $$EI\frac{d^2y}{dx^2} = \frac{Wx}{2}.$$

Integrating once, $$EI\frac{dy}{dx} = \frac{Wx^2}{4} + C.$$

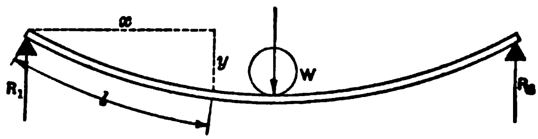


Fig. 66.

To find C we must know the slope at some point. At the middle of the beam the tangent to the elastic curve is horizontal, consequently there is no slope at that point, hence

$$\frac{dy}{dx} = 0 \quad \text{when} \quad x = \frac{L}{2}, \quad \text{therefore } C = -\frac{WL^2}{16}.$$

Then $$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16}.$$

Integrating the second time, we get

$$EIy = \frac{Wx^3}{12} - \frac{WL^2x}{16} + C'.$$

At the supports there is no deflection, therefore

$$y = 0 \quad \text{when} \quad x = 0 \therefore C' = 0.$$

Then $$EIy = \frac{Wx^3}{12} - \frac{WL^2x}{16}.$$

The deflection is a maximum when $x = \frac{L}{2}$, therefore

$$EIy_{\max} = \frac{WL^3}{96} - \frac{WL^3}{32} = -\frac{WL^3}{48}.$$

Then $$y_{\max} = -\frac{WL^3}{48\,EI}.$$

Example II. — Cantilever with a load W concentrated at the end, Fig. 67.

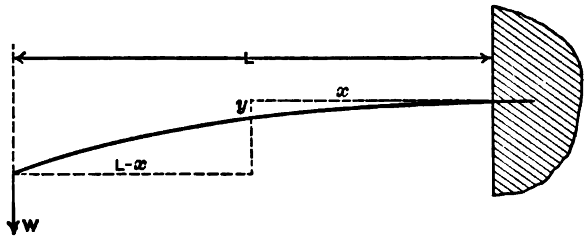


Fig. 67.

For any section of the beam we have

$$M = -W(L-x).$$

Then
$$EI\frac{d^2y}{dx^2} = -W(L-x),$$

$$EI\frac{dy}{dx} = \frac{Wx^2}{2} - WLx + C.$$

There is no slope at the wall, therefore $\frac{dy}{dx} = 0$ when $x = 0$,

$\therefore \quad C = 0.$

Then
$$EI\frac{dy}{dx} = \frac{Wx^2}{2} - WLx,$$

$$EIy = \frac{Wx^3}{6} - \frac{WLx^2}{2} + C'.$$

There is no deflection at the wall, therefore $y = 0$ when $x = 0$,

$\therefore \quad C' = 0.$

Then
$$EIy = \frac{Wx^3}{6} - \frac{WLx^2}{2}.$$

The deflection is a maximum when $x = L$, therefore

$$EIy_{\max} = \frac{WL^3}{6} - \frac{WL^3}{2} = -\frac{WL^3}{3},$$

$$y_{\max} = -\frac{WL^3}{3EI}.$$

Example III. — Simple beam uniformly loaded with w pounds per unit of length, Fig. 68. For any section of the beam we have

$$M = \frac{wLx}{2} - \frac{wx^2}{2}.$$

Then
$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2},$$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C.$$

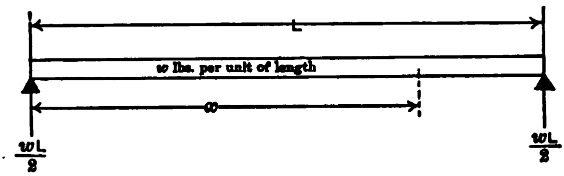


Fig. 68.

The tangent to the elastic curve is horizontal at the middle of the beam, therefore $\frac{dy}{dx} = 0$ when $x = \frac{L}{2}$. For $x = \frac{L}{2}$, we have

$$C = \frac{wL^3}{48} - \frac{wL^3}{16} = -\frac{wL^3}{24}.$$

Then
$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24},$$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} + C'.$$

$y = 0$ when $x = 0$, consequently $C' = 0$.

Then
$$y = \frac{w}{24\,EI}(2\,Lx^3 - x^4 - L^3x).$$

The deflection is a maximum at the middle of the beam, where $x = \frac{L}{2}$.

Then $y_{max} = \frac{w}{24\,EI}\left(\frac{L^4}{4} - \frac{L^4}{16} - \frac{L^4}{2}\right) = -\frac{5\,wL^4}{384\,EI} = -\frac{5\,WL^3}{384\,EI}$, in which $W = wL$, the whole load.

Example IV. — Cantilever uniformly loaded with w pounds per unit of length, Fig. 69.

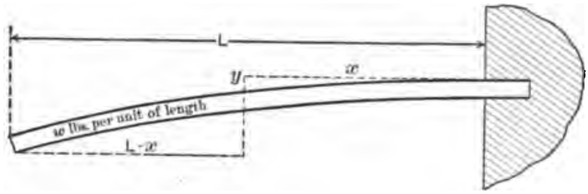


Fig. 69.

For any section of the beam

$$M = -\frac{w}{2}(L - x)^2.$$

Then $$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(L^2 - 2\,Lx + x^2),$$

$$EI\frac{dy}{dx} = -\frac{w}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) + C.$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0, \text{ consequently } C = 0.$$

Then $$EI\frac{dy}{dx} = -\frac{w}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right),$$

$$EIy = -\frac{w}{2}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) + C'.$$

$$y = 0 \text{ when } x = 0, \text{ consequently } C' = 0.$$

Then $$y = -\frac{w}{24\,EI}(6\,L^2x^2 - 4\,Lx^3 + x^4).$$

The deflection is a maximum when $x = L$, therefore

$$y_{max} = -\frac{wL^4}{8\,EI} = -\frac{WL^3}{8\,EI}, \text{ in which } W = wL \text{ is the whole load.}$$

Example V. — Beam built-in or *fixed* at the ends and uniformly loaded, Fig. 70.

This is a condition not heretofore encountered. The beam is fixed at the ends in the sense that the parts of the beam built

in are constrained to remain horizontal while the beam is being bent, but it is understood that the beam is free endwise. The reaction of the wall in keeping the built-in part horizontal introduces the couple, P, P.

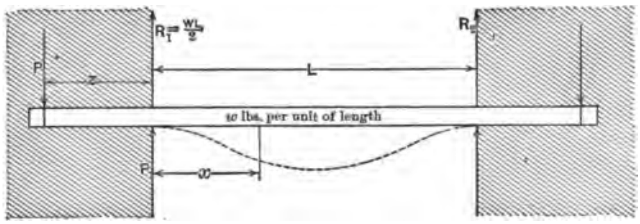


Fig. 70.

The bending moment at any section between the left support and the middle, and distant x from the left support, is

$$M = R_1x + Px - P(x + z) - \frac{wx^2}{2}$$

$$= \frac{wLx}{2} + Px - Px - Pz - \frac{wx^2}{2}$$

$$= \frac{w}{2}(Lx - x^2) - Pz.$$

Then
$$EI\frac{d^2y}{dx^2} = \frac{w}{2}(Lx - x^2) - Pz,$$

$$EI\frac{dy}{dx} = \frac{w}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) - Pzx + C.$$

From the nature of the support there is no deflection at the wall, hence $\frac{dy}{dx} = 0$ when $x = 0$, consequently $C = 0$. To find the value of Pz, which is the bending moment at the support, we must know the value of $\frac{dy}{dx}$ at some other section. At the middle of the beam the tangent to the elastic curve is horizontal,

therefore $\frac{dy}{dx} = 0$ when $x = \frac{L}{2}$. Substituting this value of x in the first derivative, we get

$$\frac{w}{2}\left(\frac{L^3}{8} - \frac{L^3}{24}\right) - \frac{PzL}{2} = 0, \text{ whence } Pz = \frac{wL^2}{12},$$

the bending moment at the support.

Substituting the value of Pz, and integrating the second time, we have

$$EIy = \frac{w}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) - \frac{wL^2x^2}{24} + C'.$$

At the support $y = 0$ and $x = 0$, therefore $C' = 0$; and y is a maximum when $x = \frac{L}{2}$, consequently

$$y_{max} = \frac{w}{2\,EI}\left(\frac{L^4}{48} - \frac{L^4}{192} - \frac{L^4}{96}\right) = -\frac{wL^4}{384\,EI} = -\frac{WL^3}{384\,EI},$$

in which $W = wL$ is the whole load.

Substituting the value of Pz in the expression for M, and letting $x = \frac{L}{2}$, we have

$$M = \frac{w}{2}\left(\frac{L^2}{2} - \frac{L^2}{4}\right) - \frac{wL^2}{12} = \frac{wL^2}{24},$$

showing the bending moment at the middle to be but half that at the support, hence

$$M_{max} = \frac{wL^2}{12} = \frac{WL}{12}.$$

For the points of inflection we place the second derivative equal to zero, thus:

$$\frac{w}{2}(Lx - x^2) - \frac{wL^2}{12} = 0, \text{ or } Lx - x^2 = \frac{L^2}{6}, \text{ whence } x = \frac{L(3 \pm \sqrt{3})}{6}.$$

40. Continuous Beams. — The beams heretofore considered have been either cantilevers or beams having but two supports. A beam resting on more than two supports is termed a continu-

ous beam. The chief difficulties in the treatment of continuous beams are the determination of the support reactions and the bending moments at the supports. When these are known the bending moment, shear, and deflection at any section may be determined.

When considering beams having two supports the support reactions were easily found by means of two equilibrium equations. With parallel forces there can be but two equilibrium equations, so that if a beam has more than two supports some other means must be adopted for finding the support reactions. The whole treatment of continuous beams is simplified by means of Clapeyron's theorem of three moments, and is fully set forth in any complete treatise on the mechanics of engineering. Only the special case of a beam uniformly loaded and resting on three supports equally spaced and on the same level will here be considered.

41. Beam Resting on Three Supports. — The beam of Fig. 71 is uniformly loaded with w pounds per unit of length and rests on three supports equally spaced and on the same level.

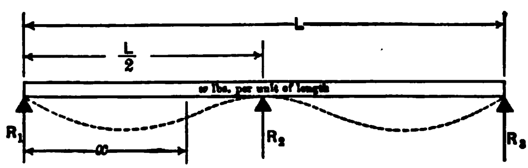


Fig. 71.

In finding the support reactions we will first assume the middle support to be removed. In such case we have found (Art. 39, Ex. III) that the deflection at the middle due to the whole load wL is $\frac{5}{384}\frac{wL^4}{EI}$.

Let W' denote the load on the middle support. Then the

upward deflection caused by the reaction at the middle support due to the concentrated load W' is, by Art. 39, Ex. I, $\dfrac{W'L^3}{48\,EI}$.

The tops of the supports being on the same level, the upward deflection must equal the downward deflection, so that

$$\frac{W'L^3}{48\,EI} = \frac{5\,wL^4}{384\,EI}, \quad \text{whence} \quad W' = \frac{5\,wL}{8}.$$

That is, the middle support bears $\frac{5}{8}$ of the whole load; and since the load is uniform, each of the end supports bears $\frac{3}{16}$ of the load.

Thus $$R_1 = R_3 = \frac{3\,wL}{16}, \qquad \text{and} \quad R_2 = \frac{5\,wL}{8}.$$

The bending moment at any section between the left and middle supports, and distant x from the left support, is

$$M = R_1 x - \frac{wx^2}{2}. \tag{13}$$

Then $$EI\frac{d^2y}{dx^2} = R_1 x - \frac{wx^2}{2},$$

$$EI\frac{dy}{dx} = \frac{R_1 x^2}{2} - \frac{wx^3}{6} + C.$$

There is no slope at the middle support, therefore

$$\frac{dy}{dx} = 0 \quad \text{when} \quad x = \frac{L}{2},$$

consequently

$$C = \frac{wx^3}{6} - \frac{R_1 x^2}{2} = \frac{wL^3}{48} - \frac{3\,wL^3}{128} = -\frac{wL^3}{384}.$$

Then $$EI\frac{dy}{dx} = \frac{R_1 x^2}{2} - \frac{wx^3}{6} - \frac{wL^3}{384},$$

$$EIy = \frac{R_1 x^3}{6} - \frac{wx^4}{24} - \frac{wL^3 x}{384}, \tag{14}$$

the constant of integration being zero, because $y = 0$ when $x = 0$.

From equation (14) the deflection, y, may be determined for any section. Thus, at the section midway between the left and middle supports, $x = \frac{L}{4}$ and we have, since $R_1 = \frac{3\,wL}{16}$,

$$EIy = \frac{wL^4}{2048} - \frac{wL^4}{6144} - \frac{wL^4}{1536} = -\frac{wL^4}{3072},$$

$$y = -\frac{WL^3}{3072\,EI},$$

in which $W = wL$ is the whole load. This value of y is not the maximum deflection, as that does not take place at the middle of the span.

To find the point of inflection between the left and middle supports we place the expression for the bending moment within those limits [equation (13)] equal to zero, thus

$$\frac{3\,wLx}{16} - \frac{wx^2}{2} = 0, \quad \text{whence} \quad x = \frac{3\,L}{8}.$$

Hence there is a point of inflection distant three-eighths the length of the beam from the left support, or one-eighth of the length of the beam to the left of the middle support; there is another point of inflection similarly situated to the right of the middle support.

To find the bending moment at the middle support, let $x = \frac{L}{2}$ in equation (13) and we have

$$M_{R_2} = \frac{3\,wL^2}{32} - \frac{wL^2}{8} = -\frac{wL^2}{32} = -\frac{WL}{32}.$$

The greatest bending moment between the supports will be at the section where the shear passes through zero. We locate this section by placing the first x-derivative equal to zero; thus, by differentiating (13),

$$\frac{dM}{dx} = R_1 - wx,$$

$$\frac{3\,wL}{16} - wx = 0, \quad \text{whence } x = \frac{3\,L}{16};$$

that is, the maximum bending moment between the supports occurs at the section distant $\frac{3L}{16}$ from the left support. To find its value we substitute $\frac{3L}{16}$ for x in (13), and

$$M = \frac{9wL^2}{256} - \frac{9wL^2}{512} = \frac{9wL^2}{512} = \frac{9WL}{512}.$$

This result is numerically less than that found for the bending moment at the middle support; hence the maximum bending moment is at the middle support, and we shall have for safe loading, $\frac{SI}{c} = \frac{WL}{32}$. The bending moment and shear diagrams are readily drawn, as shown in Fig. 72.

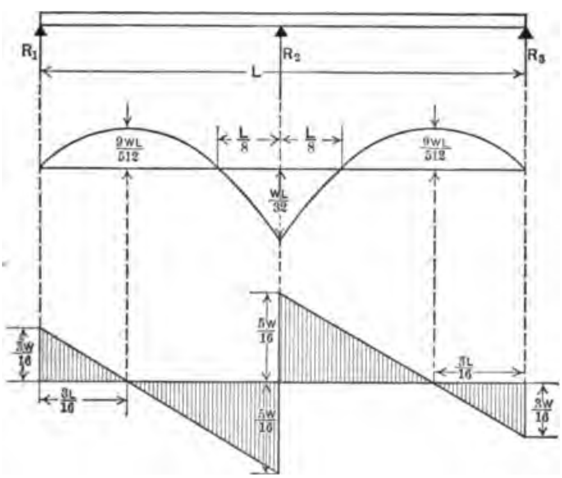


Fig. 72.

42. The Strength and Stiffness of Beams. — The direct measure of the strength of a beam is the load it will safely support, and varies inversely as the outermost fiber stress at the dangerous section. Since at the dangerous section $S = \frac{Mc}{I}$, in which M

is the maximum bending moment, it follows that $\frac{I}{Mc}$ is the expression for the comparative strength of beams. The expression for the maximum bending moments obtained in preceding pages is $M = \frac{WL}{m}$, in which m has the values 1, 2, 4, 8, and 12, depending on the conditions of support and loading. The expression for the strength of a beam may then be written $\frac{mI}{WLc}$; that is, the strength of a beam varies directly as the section modulus and inversely as the length. In the special case of beams of rectangular section, where $I = \frac{bh^3}{12}$ and $c = \frac{h}{2}$, we have $\frac{bh^2}{6}$ as the value of the section modulus $\frac{I}{c}$, so that the strength of such beams varies directly as the breadth and directly as the square of the depth. Doubling the breadth of a beam of rectangular section, therefore, doubles its strength; doubling the depth increases its strength four times. This furnishes the reason for placing rectangular beams with their greatest dimension vertical.

The ratio of the maximum deflection of a beam to its length is termed the *stiffness* of the beam, and its measure is the load the beam can carry with a given deflection. The expression for the maximum deflections obtained in this chapter is $y = \frac{WL^3}{nEI}$, in which n has the values 3, 8, 48, $\frac{384}{5}$, and 384, depending on the conditions of support and loading. From this expression we get $W = \frac{nEIy}{L^3}$; hence the stiffness of a beam varies directly as E and I, and inversely as the cube of the length. The following table furnishes a comparison of beams under different conditions of loading and support, the relative strength and stiffness of each being expressed in comparison with that of the weakest

beam, which is the cantilever with a concentrated load at the free end:

Kind of beam and nature of load and support.	Maximum bending moment.	Maximum deflection.	Relative strength.	Relative stiffness.
Cantilever, load at free end.......	WL	$\frac{WL^3}{3EI}$	1	1
Cantilever, load uniformly distributed...........................	$\frac{WL}{2}$	$\frac{WL^3}{8EI}$	2	$2\frac{2}{3}$
Simple beam, load at middle......	$\frac{WL}{4}$	$\frac{WL^3}{48EI}$	4	16
Simple beam, load uniformly distributed........................	$\frac{WL}{8}$	$\frac{5WL^3}{384EI}$	8	$25\frac{3}{5}$
Beam with fixed ends, uniformly loaded..........................	$\frac{WL}{12}$	$\frac{WL^3}{384EI}$	12	128

PROBLEMS

1. Find the expression for the maximum deflection of a beam fixed at the ends and with a load W concentrated at the middle. Find the points of inflection. *Ans.* $\frac{WL^3}{192EI}$; $\frac{L}{4}$ from each end.

2. Find the maximum deflection of a 24-inch Cambria I beam, 25 feet long and weighing 80 pounds per foot, when resting on end supports and bearing a load of 35,000 pounds at the middle. Neglect the weight of the beam. *Ans.* 0.314 in.

3. A rolled steel beam, symmetrical about its neutral axis, has a moment of inertia of 72 inch units. The beam, which is 8 inches deep, is laid across an opening of 10 feet and carries an evenly distributed load of 9 tons, including its own weight. Find the maximum deflection; find also the maximum fiber stress.
Ans. 0.218 inch; 7.5 tons per sq. in.

4. A 10-inch Cambria I beam, the moment of inertia of which is tabulated as 122.1, is laid across an opening of 16 feet. In addition to a concentrated load of 1000 pounds at the middle it carries a uniformly distributed load of 14,400 pounds, including the weight of the beam. Find the maximum deflection, taking E as 29,000,000; find also the maximum fiber stress. *Ans.* 0.416 in.; 16,120 lbs. per sq. in.

5. A beam of wood, 8 inches wide and 12 inches deep, and fixed at the ends, covers a span of 14 feet. It bears a uniformly distributed load

of 10,000 pounds, including its own weight. Find the maximum deflection, taking E as 1,500,000; find also the maximum fiber stress.

Ans. 0.072 in.; 729 lbs. per sq. in.

6. Find the safe uniformly distributed load for a 6-inch I beam resting on end supports 20 feet apart if the deflection is limited to $\frac{1}{360}$ of span. E = 30,000,000; I = 24; and weight of beam is 14.75 pounds per linear foot. *Ans.* 118.6 lbs. per linear foot.

7. How much stronger is a beam 4 inches wide, 8 inches deep, and 8 feet long, weighing 71 pounds, than one 3 inches wide, 5 inches deep, and 14 feet long, weighing 58 pounds? *Ans.* 4.9 times.

CHAPTER V

COLUMNS. SHAFTS

43. **Columns.** — A column, or strut, is a straight beam acted on compressively at its extremities, and of such length compared with its diameter, or least sectional dimension, that failure will result from buckling or lateral bending, instead of by crushing or by splitting. In addition to the many familiar applications of columns in structural work, the piston rods and connecting rods of steam engines are classed as columns.

If columns were initially absolutely straight, made of homogeneous material, and exactly centrally loaded, there would be no difference in the character of their failure from that of short specimens. These three conditions are never fulfilled in practice, and in consequence columns are weaker than short blocks of the same material.

No satisfactory theoretical discussion of columns has been made, and all the formulæ used in their design contain constants determined by experiment. The formula having the most rational basis is the one attributed to Rankine or to Gordon, which is

$$W = \frac{AS}{1 + \frac{L^2}{aK^2}},$$

in which W is the load on the column expressed in pounds, A the sectional area of the column in square inches, S the ultimate compressive strength of a short block of the material, L the length of the column in inches, K the least radius of gyration of

the section, and a a constant quantity determined by experiment for different materials.

The strength of a column depends largely upon the manner in which its ends are secured. If the ends are flat the column is said to have *square* or *flat* ends; if one end be fixed and the other end hinged, as in the case of a piston rod, the column is said to be *fixed* at one end and *free* at the other; if both ends are hinged, as in the case of a connecting rod, the column is said to be *free* or *round* at the ends.

The value of K^2 in Rankine's formula is found from the relation $I = AK^2$. The values for S in pounds per square inch, for a, and for suitable factors of safety for the three conditions of bearing, are given in the following table:

	Timber.	Cast iron.	Wrought iron.	Structural steel.	Hard steel.
S	8000	84,000	55,000	60,000	120,000
a (square ends)	3000	5,000	36,000	36,000	25,000
a (fixed and free)	1690	2,810	20,250	24,000	14,060
a (free ends)	750	1,250	9,000	9,000	6,250
Factor of safety (buildings)	8	8	6	4	5
Factor of safety (bridges)	10	....	10	5	7
Factor of safety (shocks)	15	....	15	10	15

It will be observed from the table that the ratios of the values of a for the *square* ends to the values for *fixed* and *free* and for *free* ends are as 1.75 to 1 and as 4 to 1 respectively, and these ratios indicate the relative strengths of long columns with those end conditions.

Example I. — A hollow cylindrical cast iron column 16 feet long, with square ends, sustains a load of 200,000 pounds when used in a building. Outside diameter, 10 inches; inside diameter, 8 inches. Is it safe?

Solution.—

$$A = 0.7854\,(100 - 64) = 28.27 \text{ square inches.}$$

$$K^2 = \frac{I}{A} = \frac{\pi\,(10{,}000 - 4096)}{64 \times 28.27} = 10.24.$$

Substituting in Rankine's formula, we have

$$S = \frac{200{,}000}{28.27}\left[1 + \frac{(16 \times 12)^2}{5000 \times 10.24}\right] = 12{,}168$$

pounds per square inch. The factor of safety, then, is $\frac{84{,}000}{12{,}168}$ $= 6.9$, showing the column to be unsafe.

Example II. — Find the safe load for a 12-inch Cambria I column weighing 31.5 pounds per foot, 16 feet long, and with square ends, the column to be used in a building.

Solution. — Referring to the Cambria handbook, we find that $A = 9.26$ square inches, and $K = 1.01$. The factor of safety is 4.

Then $$W = \frac{9.26 \times 15{,}000}{1 + \dfrac{(16 \times 12)^2}{36{,}000 \times (1.01)^2}} = 69{,}300 \text{ pounds.}$$

44. Design of Columns. — In designing a column we have given the form of its section, its length, the material of which it is to be made, the load it is to carry, and the manner of securing its ends. The problem is then to determine the necessary area of cross section. The general method of procedure is to determine from the data given the necessary cross-section area for a short block of the material; then, knowing that the section area of the required column must be larger, assume dimensions which give a larger area, and then, by means of Rankine's formula, ascertain the unit stress resulting from such assumption. If it is too great, the assumed area is too small; if too little, a smaller

area must be chosen. Proceed thus by trial and error until a satisfactory solution is obtained.

Example III. — A column of timber of square section, 15 feet long and with square ends, is required to support a steady load of 10 tons. Find the necessary dimensions of the section.

Solution. — Using a factor of safety of 8, we have $\frac{8000}{8} = 1000$ pounds per square inch as the safe working unit stress.

Then
$$\frac{2000 \times 10}{1000} = 20$$
square inches area of section needed for a very short column, or one less in length than ten times the least dimension of the section. As the required section must be larger, we will assume an area of 36 square inches.

Then
$$K^2 = \frac{I}{A} = \frac{(6)^4}{12 \times 36} = 3.$$
Therefore
$$S = \frac{20,000}{36}\left[1 + \frac{(15 \times 12)^2}{3000 \times 3}\right] = 2555 \text{ pounds per square inch.}$$

Since this is much larger than the allowable unit stress of 1000 pounds, we must assume a larger area.

Try an area of 64 square inches. Then
$$K^2 = \frac{(8)^4}{12 \times 64} = \frac{16}{3},$$
$$S = \frac{20,000}{64}\left[1 + \frac{(15 \times 12)^2}{3000 \times \frac{16}{3}}\right] = 945 \text{ pounds per square inch.}$$

This result is a trifle small, so we will try a square section whose side is 7.9 inches. Then
$$K^2 = \frac{(7.9)^4}{12\,(7.9)^2} = 5.2,$$
$$S = \frac{20,000}{(7.9)^2}\left[1 + \frac{(15 \times 12)^2}{3000 \times 5.2}\right] = 986 \text{ pounds per square inch,}$$

which is quite near the allowable unit stress; therefore a column of square section, having a side of 7.9 inches, will suffice.

Sometimes all the dimensions may be assumed except one, and then, after expressing A and K^2 in terms of the unknown dimension, we can substitute them in Rankine's formula and solve the resulting bi-quadratic for the unknown dimension. Thus, in the problem just solved, let x denote the length of the side of the unknown square section. Then

$$A = x^2, \quad \text{and} \quad K^2 = \frac{x^4}{12\,x^2} = \frac{x^2}{12},$$

and

$$1000 = \frac{20{,}000}{x^2}\left[1 + \frac{(15 \times 12)^2 \times 12}{3000\,x^2}\right] = \frac{20{,}000}{x^2}\left(1 + \frac{129.6}{x^2}\right),$$

whence $x = 7.9$ inches.

Example IV. — A hollow cylindrical cast iron column 20 feet in length is required to sustain a steady load of 164,000 pounds. Determine its cross-sectional dimensions.

Solution. — Using a factor of safety of 8, we have $\frac{84{,}000}{8} = 10{,}500$ pounds per square inch as the safe working unit stress.

Then $\frac{164{,}000}{10{,}500} = 15.62$ square inches needed for a very short column.

Assuming an area of cross section of 25 square inches, and assuming further that the outside diameter of the column shall be 10 inches, we shall have, calling d the inside diameter,

$$25 = 0.7854 \times 100 - 0.7854\,d^2,$$

whence $d = 8.256$ inches.

Then
$$K^2 = \frac{I}{A} = \frac{\pi\left[(10)^4 - (8.256)^4\right]}{64 \times 25} = 10.51,$$

and

$$S = \frac{164{,}000}{25}\left[1 + \frac{(20 \times 12)^2}{5000 \times 10.51}\right] = 13{,}750 \text{ pounds per sq. in.,}$$

which is greater than the allowable unit stress of 10,500 pounds. The cross-sectional area must therefore be larger, so we will assume an area of 33.75 square inches. Then

$$33.75 = 0.7854 \times 100 - 0.7854\, d^2,$$

whence $$d = 7.55 \text{ inches},$$

and $$K^2 = 9.82.$$

Then

$$S = \frac{164{,}000}{33.75}\left[1 + \frac{(20 \times 12)^2}{5000 \times 9.82}\right] = 10{,}560 \text{ pounds per square inch},$$

a result sufficiently near the allowable unit stress to warrant making the column 10 inches in outside diameter and the metal 1.25 inches thick.

Example V. — Using a factor of safety of 4, what safe load can be supported by a 7-inch Cambria channel-and-plate column 14 feet long and weighing 34.8 pounds per foot?

Solution. — From the Cambria handbook the section area of the column is found to be 10.2 square inches, and the least radius of gyration 2.63. Then

$$W = \frac{AS}{1 + \frac{L^2}{aK^2}} = \frac{10.2 \times \frac{50{,}000}{4}}{1 + \frac{(14 \times 12)^2}{36{,}000 \times (2.63)^2}} = \frac{10.2 \times 12{,}500}{1 + \frac{28{,}224}{36{,}000 \times 6.9}}$$

$$= 114{,}500 \text{ pounds.}$$

PROBLEMS

1. A hollow cylindrical cast iron column with square ends, whose thickness of metal is 2 inches, length 24 feet, and outside diameter 24 inches, is to be used in a building. What safe load can it sustain?

Ans. 1,141,000 lbs.

2. A round, solid, cast iron column with square ends is 15 feet long and 6 inches in diameter. What safe load can it carry?

Ans. 76,500 lbs.

3. A 12-inch Cambria channel-and-plate column used in a building is 18 feet long and weighs 64.8 pounds per foot. What safe load can it carry? *Ans.* 223,300 lbs.

4. Find the external diameter and thickness of metal of a hollow cast iron column that will safely carry a load of 194,000 pounds, its length being 16 feet, and the ratio of the inside and outside diameters being 0.8. Use a factor of safety of 8.

Ans. 10.5 ins.; 1.1 ins.

5. Find the thickness of metal of a hollow column of structural steel, 6 inches in diameter and 28 feet long, to support a load of 100,000 pounds in a building. *Ans.* 0.775 in.

45. **Shafts.** — A shaft in mechanics is a revolving bar used for the transmission of power generated by an engine or other motor. In the design of a shaft the important consideration is the nature of the stress to which it is to be subjected. In the form of shaft commonly known as an axle, where the load is applied transversely, the stress is chiefly due to bending; in transmission shafting, such as that used in machine shops and factories, the stress is principally due to torsion; in heavier forms, such as crank shafts, the stress is one of combined torsion and bending.

Since rotating cylindrical bodies have equal strength in all positions, shafts are usually circular in section. We shall be concerned only with those shafts of circular section that are subjected chiefly to a torsional stress, and with those subjected to a combined stress of torsion and bending.

46. **Torsion.** — When shafts are used for the transmission of power they are subjected to a shearing stress due to a *torque* or *twisting moment*, the measure of which is the product of the applied force and the distance of the point of its application from the axis of the shaft. If P denotes the applied force and r the perpendicular distance of its point of application from the axis of the shaft, then Pr is the measure of the twisting moment.

When a uniform circular shaft is subjected to a twisting moment each element parallel to the axis, such as AB of Fig. 73, undergoes a helical deformation AD, and each elemental area

of a section is subjected to a shearing stress produced by a strain called *torsion*, the stress varying directly as the distance of the elemental area from the axis. The angle θ through which a longitudinal element is twisted is called the ***angle of shear***, and varies directly as the distance of the element from the axis. The angle α is called the ***angle of twist***, and is proportional to the length of the shaft.

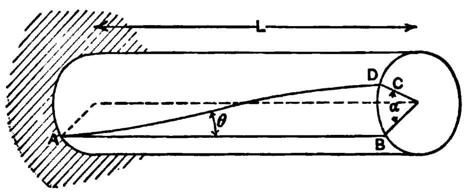


Fig. 73.

Suppose a section of length dx, Fig. 74, to be cut from a shaft by planes perpendicular to the axis, and let $d\alpha$ be the angle of twist for this section. The angle of twist varying with the length of the shaft, we shall have

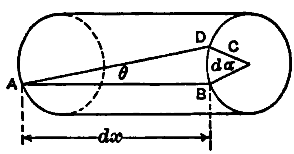


Fig. 74.

$$\frac{dx}{L} = \frac{d\alpha}{\alpha}, \quad \text{whence} \quad d\alpha = \frac{\alpha dx}{L}.$$

Expressing θ and $d\alpha$ in radians, we have

$$BD = \theta dx = c d\alpha, \quad \text{whence} \quad \theta = \frac{c d\alpha}{dx} = \frac{c\alpha}{L},$$

c being the radius of the shaft.

From Art. 30, p. 62, we have

$$\frac{\text{Unit stress}}{\text{Unit deformation}} = \frac{S}{\theta} = G,$$

whence

$$S = G\theta = \frac{Gc\alpha}{L},$$

S being the unit stress at the outermost surface.

Suppose the section of the shaft, Fig. 75, to be made up of a great number of small areas, a, a_1, a_2, etc., distant y, y_1, y_2, etc., from the axis. Each of these small areas is subjected to a shearing stress s, s_1, s_2, etc.

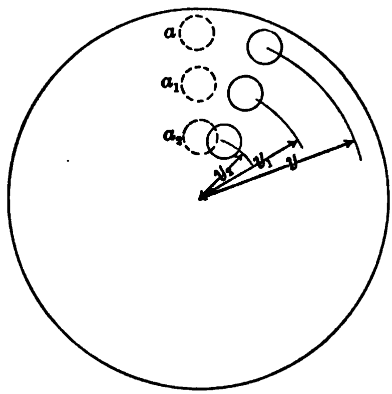


Fig. 75.

If S denotes the unit stress of the material at the outermost fiber, distant c from the axis, it is the maximum unit stress to which the material will be subjected, since the stresses vary directly as the distance from the axis.

The resisting moment of the section will be the sum of the moments of the elemental resistances, as, a_1s_1, a_2s_2, etc., and since there must be equilibrium between the internal and external forces, the resisting moment will equal the twisting moment and we shall have

$$M_t = asy + a_1s_1y_1 + a_2s_2y_2 + \text{etc.}$$

But $\frac{S}{c} = \frac{s}{y} = \frac{s_1}{y_1} = \frac{s_2}{y_2}$, whence $s = \frac{Sy}{c}$, $s_1 = \frac{Sy_1}{c}$, $s_2 = \frac{Sy_2}{c}$.

Then
$$M_t = \frac{Say^2}{c} + \frac{Sa_1y_1^2}{c} + \frac{Sa_2y_2^2}{c} + \text{etc.},$$
$$= \frac{S}{c}(ay^2 + a_1y_1^2 + a_2y_2^2 + \text{etc.}) = \frac{SI_p}{c},$$

in which I_p is the polar moment of inertia.

Then $$S = \frac{M_t c}{I_p} = \frac{Gc\alpha}{L},$$

whence $$\alpha = \frac{M_t L}{G I_p} \text{ radians.}$$

For a circular section $I_p = \frac{\pi D^4}{32}$. Hence

$$\text{Angle of twist in degrees} = \frac{32\, M_t L \times 57.3}{\pi D^4 G} = \frac{584\, M_t L}{G D^4},$$

in which D is the diameter of the shaft.

For a hollow circular shaft of external diameter D and internal diameter d, $I_p = \frac{\pi (D^4 - d^4)}{32}$, and

$$\text{Angle of twist for hollow shaft} = \frac{584\, M_t L}{G (D^4 - d^4)}.$$

From the equation $M_t = \frac{S I_p}{c}$, we get

$$\frac{M_t}{S} = \frac{I_p}{c} = Z_p,$$

in which Z_p denotes the polar modulus of the shaft section. It is thus seen that the strength of a shaft to resist torsion varies as its polar section modulus, just as the strength of a beam to resist bending varies as its plane section modulus (see Art. 34, p. 66).

From what has preceded it is known that the least plane moment of inertia, I, of a circle is just one-half its polar moment of inertia, I_p. That I_p should be greater than I might have been expected, since the material of a circular shaft is in far better form to resist torsion than to resist bending.

For shafts of small diameter and comparatively long lengths, such as line shafting in mills and shops, the angle of twist may be too great to insure safe transmission. In such cases the diameter of the shaft should be determined with reference to the

stiffness of the material rather than to its strength, the determining factor being the angle of twist.

The angle of twist should enter into the design of shafts under ten inches in diameter, and it may be accepted as a result of experience that if the angle of twist does not exceed 1° in 18 diameters of length, shafts are sufficiently stiff.

We have shown that $S = \frac{Gc\alpha}{L}$, in which α is expressed in radians, and $c = \frac{D}{2}$. With the limiting value of $L = 18\,D$ for 1° of twist, we shall have

$$S = \frac{G \times \frac{D}{2}}{57.3 \times 18\,D} = \frac{G}{2063}.$$

Taking G as 10,500,000 for wrought iron and as 12,500,000 for steel, we have for light shafting:

For wrought iron, S = 5000 pounds per square inch.

For steel, S = 6000 pounds per square inch.

Example. — A steel shaft 2 inches in diameter and 10 feet long is subjected to a twisting moment of 7500 pounds-inches. What is the angle of twist? Is the shaft safe?

Solution. —

$$\text{Angle of twist} = \frac{584\,M_tL}{GD^4} = \frac{584 \times 7500 \times 10 \times 12}{12{,}500{,}000 \times 16} = 2.63°.$$

$$\text{Allowable angle of twist} = \frac{\text{Length}}{18 \text{ diameters}} = \frac{10 \times 12}{18 \times 2} = 3.3°.$$

The shaft is therefore safe.

PROBLEMS

1. A 3-inch steel shaft, 200 feet in length, is subjected to a twisting moment of 35,000 pounds-inches. Is it safe? *Ans.* Unsafe.

2. If the shaft of problem 1 is unsafe, what should be its diameter to insure safety? *Ans.* 3.07 ins.

47. Combined Torsion and Bending. — In cases of heavy shafting, such as propeller shafts of vessels, the weight of the shaft occasions a bending action which increases the stress in the material over that due to torsion alone. The effect of this combined action of torsion and bending is not exactly determinable in practice, owing to the inexact computation of the bending moment involved. It may be shown, however, that if M and M_t denote the bending and twisting moments respectively at any section of a shaft, and M_e the equivalent twisting moment that would produce the same intensity of stress as that due to the combined torsion and bending, we shall have

$$M_e = M + \sqrt{M^2 + M_t^2}.$$

The mean twisting moment of shafts of steam engines is determined from the mean pressure in the cylinder, and the greater the ratio of expansion of the steam the greater difference there will be between the maximum and minimum moments to which the shaft will be subjected; and since a shaft must be strong enough to resist the maximum stress to which it may be liable, the twisting moment expressing the maximum stress due to the combined twisting and bending should be the basis of the calculation to determine its diameter. For practical purposes the equivalent twisting moment may be taken as 1.5 times the mean twisting moment, and in some exceptional cases this multiplier may be increased to 2.

48. Transmission of Power by Shafts. — If P denotes the mean force acting on the crank pin of an engine at a distance of r inches from the axis of the shaft, then Pr is the mean twisting moment of the shaft in pounds-inches.

For N revolutions of the shaft per minute the path of P is $2\pi rN$ inches, and the work performed is $\frac{2\pi rNP}{12}$ foot pounds.

Then

$$\text{Horse power transmitted} = \text{H.P.} = \frac{2\pi NPr}{12 \times 33{,}000},$$

whence $\quad Pr = M_t = \dfrac{12 \times 33{,}000 \times \text{H.P.}}{2\pi N} = \dfrac{63{,}025 \times \text{H.P.}}{N}$.

We have shown that

$$M_t = \frac{SI_p}{c} = \frac{S\pi D^3}{16} = 0.196\, SD^3,$$

in which D is the diameter of the shaft.

For short and heavy shafts where the angle of twist is not important, the value of S may be taken as 10,000 pounds per square inch for steel and as 8000 for wrought iron. Taking the equivalent twisting moment as $1.5\, M_t$, we shall have for heavy steel shafts:

$$D = \sqrt[3]{\frac{63{,}025 \times \text{H.P.} \times 1.5}{0.196 \times 10{,}000\,\text{N}}} = 3.64\sqrt[3]{\frac{\text{H.P.}}{N}},$$

and for heavy wrought-iron shafting, where $S = 8000$,

$$D = 3.92\sqrt[3]{\frac{\text{H.P.}}{N}}.$$

The light shafts of machine and wood-working shops, running at from 125 to 250 revolutions per minute, and carrying pulleys from which machines are driven, are subjected to bending as well as torsion. In such cases it is in accordance with good practice to take the equivalent twisting moment at 1.5 times the mean, and to take the value of S for steel as 6000 pounds per square inch, giving for the diameter the value

$$D = 4.32\sqrt[3]{\frac{\text{H.P.}}{N}}.$$

For hollow shafts in which the internal diameter d is one-half the external diameter D, the common practice for shafts of steamships, we have

$$Pr = M_t = \frac{SI_p}{c} = \frac{\pi S\,(D^4 - d^4)}{32\,c} = 0.184\, SD^3.$$

Therefore, for hollow shafts of steel, with $S = 10{,}000$ and the equivalent twisting moment 1.5 times the mean, we have

$$D = \sqrt[3]{\frac{63{,}025 \times \text{H.P.} \times 1.5}{0.184 \times 10{,}000\, N}} = 3.72 \sqrt[3]{\frac{\text{H.P.}}{N}}\,.$$

For hollow iron shafts, $S = 8000$, and

$$D = 4 \sqrt[3]{\frac{\text{H.P.}}{N}}\,.$$

Example. — A steel shaft is to have 30 per cent of its section area removed in making an axial hole, and is to transmit 9000 H.P. at 120 revolutions per minute. Taking the equivalent twisting moment as 1.5 times the mean twisting moment, find the internal and external diameters of the shaft; show that the shaft is 25.55 per cent lighter than a solid shaft would be to transmit the same power; and show that the two shafts would be of equal strength.

Solution. — Area of removed section $= \dfrac{0.3\,\pi D^2}{4}$.

Therefore $\dfrac{\pi d^2}{4} = \dfrac{0.3\,\pi D^2}{4}$, and $d = 0.5477\,D$.

$$\text{H.P.} = \frac{2\,\pi N P r}{12 \times 33{,}000},$$

whence

$$1.5\,Pr = M_t = \frac{9000 \times 12 \times 33{,}000 \times 1.5}{2\,\pi \times 120} = 7{,}090{,}000 \text{ pounds-inches.}$$

$$M_t = \frac{SI_p}{c} = \frac{\pi S(D^4 - d^4)}{32\,c} = \frac{10{,}000\,\pi [D^4 - (0.5477 D)^4]}{16\,D} = 1786.78 D^3.$$

Then $1786.78\,D^3 = 7{,}090{,}000,$

whence $D = 15.83$ inches,

and $d = 0.5477\,D = 8.67$ inches.

A solid steel shaft to transmit 9000 H.P. at 120 revolutions per minute would have a diameter

$$D = 3.64 \sqrt[3]{\frac{9000}{120}} = 15.35 \text{ inches.}$$

Since the weights of shafts are proportional to their sectional areas, and as the areas vary as the squares of their diameters, we find the hollow shaft to be

$$\frac{100\{(15.35)^2 - [(15.83)^2 - (8.67)^2]\}}{(15.35)^2} = 25.55 \text{ per cent}$$

lighter than the solid shaft.

For the hollow shaft

$$Z_p = \frac{I_p}{c} = \frac{M_t}{S} = 0.1787\,D^3 = 0.1787 \times (15.83)^3 = 709.$$

For the solid shaft

$$Z_p = \frac{\pi D^4}{32c} = \frac{\pi D^3}{16} = \frac{3.1416\,(15.35)^3}{16} = 709.$$

The moduli of the sections being equal, the shafts are of equal strength.

PROBLEMS

1. Find the diameter of a wrought-iron shaft to transmit 90 H.P. at 130 revolutions per minute. What should be the diameter if there is a bending moment equal to the twisting moment?

Ans. 3.54 ins.; 4.75 ins.

2. Find the diameter of a steel shaft for a steam engine having an overhung crank. Diameter of cylinder, 18 inches; mean steam pressure, 130 pounds per square inch; stroke of piston, 30 inches; overhang of crank, *i.e.* middle of crank pin to middle of bearing, 18 inches. Use $M_e = M + \sqrt{M^2 + M_t^2}$. *Ans.* 9.56 ins.

3. A 4-inch steel shaft 30 feet long is found to have an angle of twist of 6.2° when transmitting power while making 130 revolutions per minute. Find the H.P. transmitted. *Ans.* 194.64.

4. Find the diameter of a steel shaft, 50 feet long to transmit 30 H.P. at 210 revolutions a minute. *Ans.* 3.2 ins.

5. Find the diameter of a hollow steel shaft to transmit 10,000 H.P. at 120 revolutions per minute, the external diameter being twice the inner, and the equivalent twisting moment 1.5 times the mean.

Ans. 16.25 ins.

6. A steel shaft is to have 25 per cent of its sectional area removed in making an axial hole, and is then to transmit 7200 H.P. at 120 revolutions per minute. Taking the equivalent twisting moment as 1.5 times the mean, find the internal and external diameters of the shaft, and show that the shaft is 21.7 per cent lighter than a solid shaft would be to transmit the same power. *Ans.* 14.56 ins. and 7.28 ins.

7. The main shaft of a machine shop, in transmitting 40 H.P. at 120 revolutions per minute, carries the average number of pulleys. Find its diameter if made of steel. *Ans.* 3 ins.

CHAPTER VI

INTERNAL WORK DUE TO DEFORMATION. SUDDENLY APPLIED LOADS

49. Resilience. — The stresses thus far encountered have been those produced by external forces gradually applied to bodies under static conditions, and the resulting deformations necessarily produced internal work in the bodies. As the stresses produced were within the elastic limit, the bodies resumed their original shapes upon the removal of the acting forces, and in doing so the internal work due to the deformations was, in accordance with the principle of the conservation of energy, given out in the form of mechanical energy. It is thus seen that the internal work of deformation is a form of potential energy, called *resilience.*

50. Effect of Suddenly Applied Loads. — The sudden application of loads, such as a train passing rapidly over a bridge, causes greater deflections than would be the case were the same loads applied gradually. These deflections are but momentary, as resilience causes a vibratory action in the affected bodies until the effect of the shock due to the sudden application of the load disappears.

If a load W be gradually applied to a beam, producing a stress S, the intensities of the load and stress will gradually increase from 0 to W and from 0 to S respectively, the mean values being $\frac{W}{2}$ and $\frac{S}{2}$. If y be the deflection, we shall have for the equality of the external and internal work

$$\frac{W}{2} \times y = \frac{S}{2} \times y, \text{ whence } S = W.$$

If the same load be applied suddenly, its intensity remaining constant during the period of action, the resulting stress S' will increase from 0 to S', its mean value being $\frac{S'}{2}$, and a deflection y' will be produced. We shall then have

$$Wy' = \frac{S'}{2} \times y', \quad \text{whence} \quad S' = 2W.$$

In other words, the sudden application of a load produces a stress twice as great as that produced by the same load applied gradually.

Should a load W fall on a beam through a distance h and produce a deflection y, the kinetic energy developed would be $W(h + y)$; but all this energy would not be converted into work of deformation on account of the inevitable loss of kinetic energy sustained by all partly elastic bodies during impact.

Example I. — What load may fall through a distance of 8 inches on the middle of a 12-inch Cambria I beam in order that the maximum fiber stress shall not exceed 20,000 pounds per square inch, supposing 70 per cent of the kinetic energy of the falling load to be converted into work of deformation? The length of the beam is 16 feet, and its tabulated moment of inertia is 215.8.

Solution. — Let W denote the gradually applied load that would perform, in producing a deflection y, the same work of deformation as that performed by the falling load W'. The mean value of W is $\frac{W}{2}$, and its work of deformation is $\frac{Wy}{2}$.

The bending moment at the middle section due to W is $\frac{WL}{4}$, and we shall have

$$S = \frac{Mc}{I} = \frac{WLc}{4I}, \quad \text{whence} \quad W = \frac{4SI}{Lc}.$$

The deflection due to W we have found to be $\frac{WL^3}{48EI}$, (Art. 39, Ex. I).

Hence $y = \frac{WL^3}{48\,EI} = \frac{SL^2}{12\,Ec} = \frac{20,000\,(16 \times 12)^2}{12 \times 30,000,000 \times 6} = 0.341$ inch.

Then

$$\text{Work of deformation by } W = \frac{Wy}{2} = \frac{2\,SI}{Lc} \times \frac{SL^2}{12\,Ec} = \frac{S^2IL}{6\,Ec^2}$$

$$= \frac{(20,000)^2 \times 215.8 \times 16 \times 12}{6 \times 30,000,000 \times 36} = 2557.6 \text{ inch-pounds.}$$

Work of deformation by W' is $0.7\,W'\,(8 + 0.341)$, hence

$$2557.6 = 0.7\,W' \times 8.341,$$

whence $$W' = 438 \text{ pounds.}$$

PROBLEM

1. From what height may a load of 1200 pounds fall on a 15-inch Cambria I beam 15 feet long and having a moment of inertia of 511, in order that the maximum fiber stress shall not exceed 30,000 pounds per square inch, and supposing 70 per cent of the kinetic energy of the falling weight to be converted into work of deformation?

Ans. 9.37 ins.

The sudden application of loads produces impact, resulting in stresses greatly in excess of those produced by the same loads when applied gradually, and in the recovery from the shocks thus occasioned the material sustains such rapid vibrations that its molecular structure and elasticity may be impaired. In the design of machines and structures that are subjected to shocks the effect of impact is an important consideration, and the determination of the resilience of a given material is the best measure of its ability to withstand shocks.

51. Modulus of Resilience. — If a bar be placed in a testing machine and subjected to a gradually increasing load in a direction to produce a tensile stress, the elongation produced will be proportional to the load, provided the elastic limit of the material be not exceeded. The load, or external force, will gradually increase from 0 to W, and its mean value will be $\frac{W}{2}$. If y

denotes the elongation, the expression for the work done is $\frac{Wy}{2}$. It is a fundamental assumption that the total stress in the bar is uniformly distributed throughout its section, so that if A denotes the area of a section of the bar and S denotes the unit stress, we shall have $W = AS$; and it has been shown in Art. 31, p. 63, that $y = \frac{SL}{E}$, so that the expression for the work in this instance may be written

$$\text{Internal work} = \frac{S^2}{2E} \cdot AL.$$

But AL is the volume of the bar, consequently the work done is proportional to the volume of the bar, and therefore to its weight.

The work done in stretching the bar to its elastic limit is its resilience, and if S be taken as the unit stress at the elastic limit, the ratio $\frac{S^2}{2E}$ is known as the *modulus of resilience*. The resilience of the bar is then found by multiplying its volume by the modulus of resilience.

Example II.—A steel bar 10 feet long and 2 inches in diameter is stretched to the elastic limit; what is its resilience?

Solution. — The elastic limit of steel in tension is 50,000 pounds per square inch, and the coefficient of elasticity is 30,000,000 pounds per square inch.

Then, $\quad \text{Modulus of resilience} = \frac{S^2}{2E} = \frac{(50{,}000)^2}{2 \times 30{,}000{,}000} = \frac{250}{6},$

and $\quad \text{Volume of bar} = 3.1416 \times 120 = 377 \text{ cubic inches.}$

Then, $\quad \text{Resilience of bar} = \frac{250 \times 377}{6} = 15{,}708 \text{ inch-pounds.}$

PROBLEMS

1. In gradually elongating a steel bar 1 square inch in section and 20 feet long, work to the amount of 150 foot pounds is expended. Find the applied force and the amount of elongation produced.

Ans. 21,213 lbs.; 0.17 in.

2. A wrought iron bar 4 inches in diameter and 10 feet long has its unit tensile stress increased from 10,000 to 20,000 pounds per square inch. What additional potential energy is stored in the rod by the operation? *Ans.* 673 ft. lbs.

3. If 600 foot pounds of work are expended in increasing the unit tensile stress of a steel bar 4 inches in diameter from 10,000 to 20,000 pounds per square inch what is the length of the rod? *Ans.* 7.5 ft.

52. Resilience of Beams. — The work performed in bending a beam to the maximum deflection within the elastic limit is the resilience of the beam. Denoting the load by W, the stress in the beam increases from 0 to W, the mean value being $\frac{W}{2}$. The work performed, or the resilience, is therefore equal to half the load multiplied by the deflection. Several illustrations of the methods of finding the resilience of beams under different conditions of loading will here be given.

I. *Cantilever with load at extremity.*—The maximum deflection being $\frac{WL^3}{3\,EI}$ (Art. 39, Ex. II), the resilience is $\frac{WL^3}{3\,EI} \cdot \frac{W}{2} = \frac{W^2L^3}{6\,EI}$.

$$M_{\max} = WL = \frac{SI}{c}, \text{ whence } W = \frac{SI}{Lc}.$$

Then,
$$\text{Resilience} = \frac{S^2I^2L^3}{6\,EIL^2c^2} = \frac{S^2IL}{2\,E \cdot 3\,c^2} = \frac{S^2}{2\,E} \cdot \frac{K^2}{3\,c^2} \cdot AL,$$

in which AK^2 is substituted for I.

For rectangular sections, $c = \frac{d}{2}$ and $K^2 = \frac{I}{A} = \frac{bd^3}{12\,bd} = \frac{d^2}{12}$, so that $\frac{K^2}{c^2} = \frac{1}{3}$. Then, for cantilevers of rectangular section and with concentrated load at the free end, we have

$$\text{Resilience} = \frac{S^2}{2\,E} \cdot \frac{AL}{9},$$

or it is the product of the modulus of resilience and one-ninth the volume of the beam.

II. *Simple beam with load concentrated at middle.* — The maximum deflection being $\frac{WL^3}{48\,EI}$ (Art. 39, Ex. I), the resilience is $\frac{WL^3}{48\,EI}\cdot\frac{W}{2} = \frac{W^2L^3}{96\,EI}$.

$$M_{\max} = \frac{WL}{4} = \frac{SI}{c}, \text{ whence } W = \frac{4\,SI}{Lc}.$$

Then, $$\text{Resilience} = \frac{16\,S^2I^2L^3}{96\,EIL^2c^2} = \frac{S^2IL}{2\,E\cdot 3\,c^2} = \frac{S^2}{2\,E}\cdot\frac{K^2}{3\,c^2}\cdot AL.$$

For rectangular sections, $\frac{K^2}{c^2} = \frac{1}{3}$, and we shall have for a simple beam with load at middle

$$\text{Resilience} = \frac{S^2}{2\,E}\cdot\frac{AL}{9},$$

which is the same as that found for a cantilever with load at its extremity.

III. *Cantilever uniformly loaded.* — If a cantilever be loaded with w pounds per linear unit, the elemental load is wdx, and if y be the corresponding deflection the elemental external work is $\frac{wydx}{2}$, the summation of which will be the resilience.

In Art. 39, Ex. IV, we found

$$y = \frac{w}{24\,EI}(6\,L^2x^2 - 4\,Lx^3 + x^4).$$

Then, $$\text{Resilience} = \frac{w^2}{48\,EI}\int_0^L (6\,L^2x^2 - 4\,Lx^3 + x^4)\,dx$$

$$= \frac{w^2}{48\,EI}\left[2\,L^2x^3 - Lx^4 + \frac{x^5}{5}\right]_0^L = \frac{6\,w^2L^5}{240\,EI}.$$

The bending moment at any section distant x from the wall is $-\frac{w\,(L-x)^2}{2} = -\frac{w\,(L^2 - 2\,Lx + x^2)}{2}$, and is a maximum when $x = 0$, hence

$$M_{\max} = -\frac{wL^2}{2} = \frac{SI}{c}, \text{ whence } w = -\frac{2\,SI}{L^2c}.$$

Then, $$\text{Resilience} = \frac{S^2}{2E}\cdot\frac{1}{5}\cdot\frac{K^2}{c^2}\cdot AL.$$

For rectangular sections, $\frac{K^2}{c^2} = \frac{1}{3}$, hence

$$\text{Resilience} = \frac{S^2}{2E}\cdot\frac{AL}{15}$$

for cantilevers of rectangular section and uniform load.

IV. *Simple beam uniformly loaded.* — The elemental load is wdx, and if y be the corresponding deflection, the elemental external work is $\frac{wydx}{2}$, the summation of which will be the resilience.

In Art. 39, Ex. III, we found

$$y = \frac{w}{24\,EI}(2\,Lx^3 - x^4 - L^3x), \text{ therefore}$$

$$\text{Resilience} = \frac{w^2}{48\,EI}\int_0^L (2\,Lx^3 - x^4 - L^3x)\,dx$$

$$= \frac{w^2}{48\,EI}\left[\frac{Lx^4}{2} - \frac{x^5}{5} - \frac{L^3x^2}{2}\right]_0^L = -\frac{w^2L^5}{240\,EI}.$$

The bending moment at any section distant x from the left support is $\frac{wLx}{2} - \frac{wx^2}{2} = \frac{w}{2}(Lx - x^2)$, and is a maximum at the middle of the beam, where $x = \frac{L}{2}$, hence

$$M_{\max} = \frac{wL^2}{8} = \frac{SI}{c}, \text{ whence } w = \frac{8\,SI}{L^2c}.$$

Then

$$\text{Resilience} = \frac{S^2}{2E}\cdot\frac{8}{15}\cdot\frac{K^2}{c^2}\cdot AL.$$

For rectangular sections, $\frac{K^2}{c^2} = \frac{1}{3}$, hence

$$\text{Resilience} = \frac{S^2}{2E}\cdot\frac{8AL}{45}$$

for beams of rectangular section and uniformly loaded.

53. Resilience of Torsion. — Suppose the shaft of Fig. 76 to be subjected to a twisting moment $P'r'$, and in consequence suppose y to be the displacement of an elemental area a distant r from the axis. If R denotes the unit shearings tress in area a, then

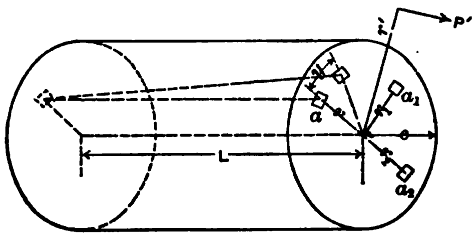


Fig. 76.

$$\text{Work performed in displacing } a = \frac{0 + Ra}{2} \times y = \frac{Ray}{2}.$$

From Art. 31 we shall have

$$\frac{G}{L} = \frac{R}{y}, \quad \text{whence } y = \frac{RL}{G}.$$

Then, $\qquad$ Work performed in displacing $a = \dfrac{R^2aL}{2G}$.

$$\text{Total work performed} = \frac{R^2aL}{2G} + \frac{R_1^2a_1L}{2G} + \frac{R_2^2a_2L}{2G} + \text{etc.},$$

which is an expression for the *resilience of torsion* of the beam.

If S denotes the unit shearing stress in the remotest elemental area, distant c from the axis, then $\dfrac{S}{c}$ is the unit shearing stress at a unit's distance from the axis, and $\dfrac{Sr}{c}$ is the unit shearing stress in the elemental area a distant r from the axis. Hence $R = \dfrac{Sr}{c}$, and in like manner $R_1 = \dfrac{Sr_1}{c}$, and $R_2 = \dfrac{Sr_2}{c}$. We shall then have

$$\text{Resilience of torsion} = \frac{S^2r^2aL}{2Gc^2} + \frac{S^2r_1^2a_1L}{2Gc^2} + \frac{S^2r_2^2a_2L}{2Gc^2} + \text{etc.}$$

$$= \frac{S^2L}{2Gc^2}(ar^2 + a_1r_1^2 + a_2r_2^2 + \text{etc.})$$

$$= \frac{S^2LI_p}{2Gc^2} = \frac{S^2}{2G} \cdot \frac{LAK^2}{c^2} = \frac{S^2}{2G} \cdot \frac{K^2}{c^2} \cdot AL,$$

since the polar moment of inertia $I_p = AK^2$.

For circular sections, $c = \frac{D}{2}$ and $K^2 = \frac{I_p}{A} = \frac{\pi D^4}{32} \div \frac{\pi D^2}{4} = \frac{D^2}{8}$,

so that $$\frac{K^2}{c^2} = \frac{1}{2}.$$

Then, for shafts of circular section we have

$$\text{Resilience of torsion} = \frac{S^2}{2G} \cdot \frac{AL}{2},$$

or, it is the product of the coefficient of resilience for torsion and one-half the volume of the shaft.

Comparing the expressions derived for the resilience of a bar in tension, of beams of rectangular sections under different conditions of loading, and of a circular shaft under torsion, we observe that the resilience of the bar is 9 times as great as that of a cantilever loaded at its free end, 9 times that of a simple beam loaded at the middle, 15 times that of a cantilever uniformly loaded, $5\frac{5}{8}$ times that of a simple beam uniformly loaded, and twice that of a shaft under torsion.

PROBLEMS

1. Derive the expression for the resilience of a beam uniformly loaded with w pounds per linear unit and fixed at the ends. *Ans.* $\frac{S^2}{2E} \cdot \frac{AL}{15}$.

2. Derive the expression for the resilience of a beam fixed at the ends and having a load W concentrated at the middle. *Ans.* $\frac{S^2}{2E} \cdot \frac{AL}{9}$.

CHAPTER VII

GRAPHIC STATICS

System of Lettering. Force Diagram. Funicular Polygon

54. **Graphic Statics.** — The methods by which static problems are solved by means of scale drawings constitute *graphic statics*.

In very many cases the determination by calculation of the forces transmitted through the different parts of a structure involve tedious and difficult processes, with the consequent liability to error, and in the end the effort to check the accuracy of the results occasions a procedure as prolonged and tedious as the processes themselves. By the graphic method, however, solutions are readily obtained, and the process itself furnishes a check as to accuracy.

55. **System of Lettering.** — To the system of lettering diagrams devised by A. H. Bow is due much of the facility in making graphic solutions.

The two important features of the Bow system of lettering are: (1) The placing of a letter in each of the spaces between the lines of action of the external forces; (2) the naming in clockwise order of each force by the two letters flanking it.

The forces of Fig. 77 will serve to illustrate the Bow system. The five forces are in equilibrium, and the letters A, B, C, D, and E are placed in the spaces between them. Other letters, or even numbers, would serve the purpose, and they might be placed in any order, but it will be found convenient to commence at the left and letter the spaces alphabetically. The force of 20 pounds is flanked by the letters A and B and is known as the

force AB, not as BA, since the letters must be read clockwise. In like manner the remaining forces are known as BC, CD, DE, and EA.

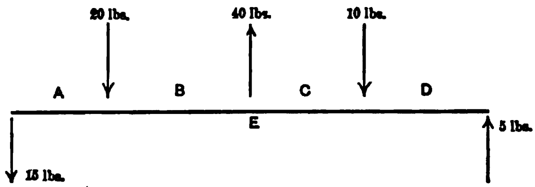


Fig. 77.

Fig. 78.

Should the system of forces act on an open-framed structure, such as a roof truss, Fig. 79, a letter must be placed within each open space of the frame in addition to those placed in the spaces between the external forces. The external forces, by the Bow notation, are known as AB, BC, CD, DE, and EA. The stress in the member connecting joints 1 and 2 is known as BG. The stress in the member separating the spaces lettered F and G, may be known as GF or as FG, according to the end of the member under consideration. If the end considered is that at joint 1 then the stress in the member is known as GF, because the forces about that joint, taken in clockwise order, are AB, BG, GF, and FA. If the other end of the member is under consideration the stress is known as FG, because the forces about the joint at that end, in clockwise order, are known as FG, GH, HI, IE, and EF.

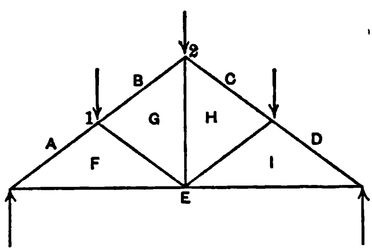


Fig. 79.

56. Force Diagram. — Taking the forces of Fig. 77 in clockwise order, and denoting them by the corresponding small letters of the alphabet, we may represent them in magnitude and direction in a diagram by lines drawn to some chosen scale. Such a diagram is known as a *force diagram*. Thus, A being the first letter in clockwise order from the left, the letter a will be the starting point of the force diagram of Fig. 78, and since the force AB acts downward, ab, one inch in length, will represent it to the scale of 20 pounds to the inch. The force BC, next in clockwise order and equal to 40 pounds, acts upward and will be denoted by bc, 2 inches in length and measured upward. In like manner, cd measured downward and one-half inch in length, denotes the force CD of 10 pounds; de measured upward and one-quarter inch in length, denotes the force DE of 5 pounds; and, finally, ea, which is found to measure three-quarters of an inch, properly denotes the force EA of 15 pounds. It will be noted that the force EA is of just sufficient magnitude to fill, when drawn to scale, the space between e and a and thus close the force diagram. This force diagram is, in reality, a closed polygon which has resolved itself into a straight line in consequence of all the forces being parallel. Had it not closed, the system being in equilibrium, there would have been an error in the construction.

With a correct construction, and the last point of a force diagram not falling on the first point, the construction would indicate a resultant force, equal in magnitude to the scale distance between the last point and the first point, and acting upward or downward according as the last point had fallen above or below the first point.

If one of the forces of Fig. 77 were unknown it could be found by means of the force diagram. Suppose the force BC unknown. Then, commencing with the force CD, the force diagram would be constructed by representing in clockwise order the forces CD, DE, EA, and AB by the scale distances cd, de, ea, and ab respec-

tively, of Fig. 78, *c* being the first point of the diagram and *b* the last thus determined. The missing force must then be represented by *bc*, measured upward, and as it measures 2 inches, it correctly does so for the force *BC* of 40 pounds.

The system of lettering enables the resultant of any number of the forces to be read at once from the force diagram. Suppose the resultant of the forces *BC*, *CD*, and *DE* were required. The first and last letter in the naming of these forces are *B* and *E* respectively, so that *be* on the force diagram represents the required resultant in magnitude and direction. The measurement of *be* is found to be 1.75 inches, which, to the scale, represents 35 pounds, the resultant of the forces 40 + 5 − 10, and it acts upward, since *be* is measured upward.

57. Funicular Polygon. — Suppose a jointed frame, Fig. 80, to be acted on at its hinged joints by a system of forces in equilibrium, the members, bars, or links of the frame being free to adjust themselves to the best position for withstanding the action of the forces. Such a figure is called a funicular polygon, the word *funicular* having no mechanical significance.

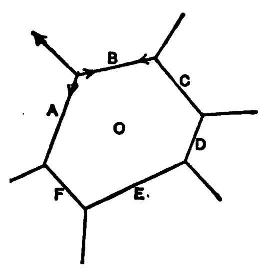


Fig. 80.

Since the system is in equilibrium, the funicular polygon must, of course, be a closed polygon. The equilibrium is occasioned by the balance between the internal forces, or stresses, in the members and the external forces, and the whole being in equilibrium, each joint in itself is in equilibrium.

Since the equilibrium at each joint is the result of the action of three concurring forces, viz., the external force at the joint and the stresses set up in the two members, it follows that a triangle may be constructed for each joint which will represent these forces in magnitude and direction.

If, for example, the force *AB* be known, the equilibrium of the

joint at which it is applied is maintained by the action of the external force AB and the stress forces in the members BO and OA. Draw ab, Fig. 81, parallel to the force AB, and make its length, to some chosen scale, represent the magnitude of AB. From b and a draw lines parallel to BO and OA respectively, intersecting at o. Then abo is the triangle of forces for the joint ABO, and bo and oa represent not only the directions of the stresses in the members BO and OA, but their magnitudes as well, to the same scale as ab represents the force AB. The stress bo in the end of the member BO at which AB is applied occasions an equal and opposite stress ob at the other end of BO, and, in fact, such is the case in all of the members of the polygon, for in no other way could the equilibrium be maintained. Taking the next joint in clockwise order, we have the external force BC and the stress forces CO and OB in equilibrium. But ob has just been found to represent in magnitude and direction the stress OB. Hence, by drawing from b and o lines parallel respectively to BC and CO, intersecting at c, we shall have bco as the triangle of forces for the joint BCO, and bc and co will represent in magnitude and direction the force BC and the stress in the member CO respectively. We now know the stress oc at the joint CDO, and can construct the triangle of forces, cdo, for that joint. In like manner we can proceed and determine all the external forces and stresses in the members, the last line, fa, closing the diagram, thus proving that the system is in equilibrium.

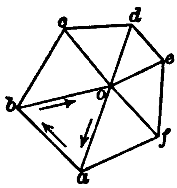


Fig. 81.

It will be observed that the sides of the polygon just constructed represent the external forces of the funicular polygon, and therefore $abcdef$ is the force diagram. Furthermore, all the lines representing the stresses in the members of the funicular

polygon meet at a point called the *pole*. These stress lines are known as *vectors*.

From what has preceded it is seen that if all the external forces that are applied to a funicular polygon are known in magnitude and direction, and also the directions of two of its members, the force polygon can be drawn. For the force diagram can be drawn from the known forces, and the intersection of the vectors parallel to the known directions of the two members gives the pole. The directions of the remaining members are then found by drawing the rest of the vectors.

A force diagram of any system of forces in equilibrium can be drawn, and by choosing any pole, a funicular polygon with respect to that pole can then be constructed to which the forces may be applied.

If the magnitudes and directions of a system of forces acting on a body are known and the system is not in equilibrium, the line of action of the force required for equilibrium may be determined by means of the funicular polygon. For the force diagram of the given forces may be drawn and the gap representing its lack of closure will, by the polygon of forces, give the magnitude and direction of the resultant. A pole for this force diagram may be selected arbitrarily and vectors drawn. Starting at a selected point in the line of action of any one of the forces, a funicular polygon may be drawn with respect to the chosen pole, and the intersection of the two members of this funicular that are parallel to the vectors drawn to the extremities of the resultant in the force diagram gives the joint of the funicular at which the required force, equal and opposite to the resultant, must be applied.

PROBLEMS

1. The parallel forces of Fig. *a* are in equilibrium. Find by a force diagram the magnitude of the force *BC*. *Ans.* 17.

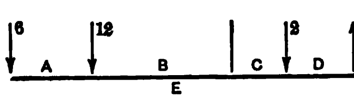
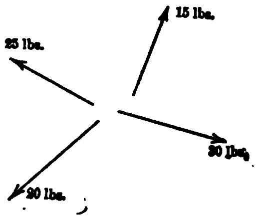


Fig. *a*.



2. Forces of 20 pounds, 25 pounds, 15 pounds, and 30 pounds act on a body in the directions shown in Fig. *b*. Find the magnitude and direction of the resultant. Find also by means of a funicular polygon the line of action of the force required for equilibrium.

Ans. Resultant = 11 lbs.

Fig. *b*.

58. Illustrations. — To illustrate properties of the funicular polygon the closed polygon of Fig. 82 has been drawn, having the known forces AB, BC, CD, DE, and EA acting at its joints.

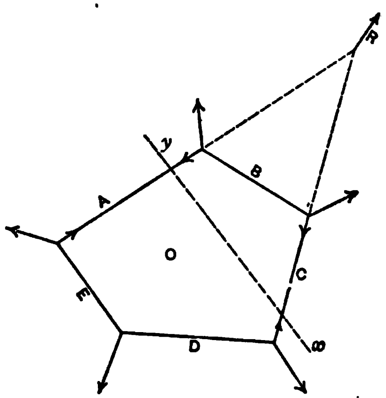


Fig. 82.

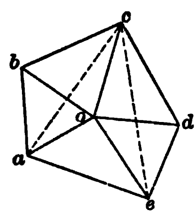


Fig. 83.

Draw to a selected scale the force diagram *abcde* of Fig. 83. The triangle of forces *abo* for the joint ABO determines the pole *o*, and the other vectors may then be drawn.

Let a plane *xy* divide the polygon into two parts. The forces of the part to the right of *xy* are AB and BC, and, by the triangle of forces, their resultant is *ac*, Fig. 83, acting from *a* to *c*. The forces of the part to the left of *xy* are CD, DE, and EA, and, by the polygon of forces, their resultant is *ca*, acting from *c* to *a*. Hence, the resultant of the forces of one part has the same magnitude and line of action as the resultant of the forces of the

other part, but acting in opposite directions, showing that the resultant of the forces of one part maintains equilibrium with the forces of the other part.

To find where the resultant of the two forces to the right of *xy* acts, we replace the forces *ab* and *bc* in the force diagram by their resultant *ac*, so that our force diagram now becomes *acde*. The vectors *oc*, *od*, *oe*, and *oa* of the force diagram have *o* as their pole, so that a funicular polygon may be drawn with respect to *o*, having its sides parallel to these vectors. We already have *OC*, *OD*, *OE*, and *OA* of Fig. 82 parallel to these vectors, but they do not close the polygon, and since a funicular polygon must close we produce *OC* and *OA* until they intersect, and at the joint thus formed the resultant *R*, having the magnitude and direction of *ac*, will act as shown by the dotted lines of Fig. 82. By a similar process the resultant, *ca*, of the forces *CD*, *DE*, and *EA* may be shown to act at the same joint but in the opposite direction.

An inspection of the funicular polygon of Fig. 82 and of the force diagram of Fig. 83 shows:

(*a*) The resultant of the forces *CD*, *DE*, and *EA* to the left of the section *xy* is given by *ca*, the first and last letters of the forces when named in clockwise order; in like manner the resultant of the forces *AB* and *BC* to the right of *xy* is *ac*.

(*b*) The letters in the force diagram which name the resultant also name the members of the funicular which have to be produced to their intersection in order to get a point in the line of action of the resultant. Thus, the resultant of the forces *CD* *DE*, and *EA* is *ca*, and by producing the members *C* and *A* to their intersection in Fig. 82 a point in the line of action of the resultant is obtained.

It will be observed that the introduction of the external force *R*, the resultant of the forces *AB* and *BC*, changes the original funicular polygon to one having fewer joints by one. Should the

members C and E be produced to their point of intersection and the resultant ce of the forces CD and DE be applied at the point, the funicular would be reduced to one of three joints.

The members cut by the section xy are A and C, and the stresses in these members, in magnitude and direction, are oa and oc of Fig. 83. On the right side of xy the stresses in the members A and C act in the directions shown by the arrowheads, and their resultant, in magnitude and direction, is ca, which is opposed by the equal and opposite external force R, or ac. On the left side of xy the stresses in the members A and C act in the directions indicated by the arrowheads, and their resultant is ac in magnitude and direction. The external forces to the left of xy are CD, DE, and EA, and their resultant in magnitude and direction is ca, which is opposed by the equal and opposite resultant ac of the stresses in the members A and C. It is thus seen that on either side of the section xy there is equilibrium between the external forces and the internal stresses in the members cut by xy, and on this principle is founded the section method of determining stresses, to be referred to later.

A practical application of the funicular polygon will be made by taking the beam of Fig. 27, p. 31, reproduced in Fig. 84. The linear scale is 1 inch = 4 feet, and the load scale 1 inch = 150 pounds. Then, W_1 = 60 pounds = 0.4 inch to scale, W_2 = 45 pounds = 0.3 inch, and W_3 = 90 pounds = 0.6 inch. Letter the beam according to the Bow system, so that W_1, W_2, W_3, R_2, and R_1 will be known as AB, BC, CD, DE, and EA respectively.

To construct the force diagram we set off, vertically downward, ab equal in length to 0.4 inch to represent the downward force W_1, or AB, to scale; bc equal in length to 0.3 inch to represent W_2, or BC; and cd equal in length to 0.6 inch to represent W_3, or CD. Then we know that da is the closing line of the force

diagram, all the forces being vertical, and that it represents the sum of the reactions R_1 and R_2; but we do not know the amount of the load borne by each support.

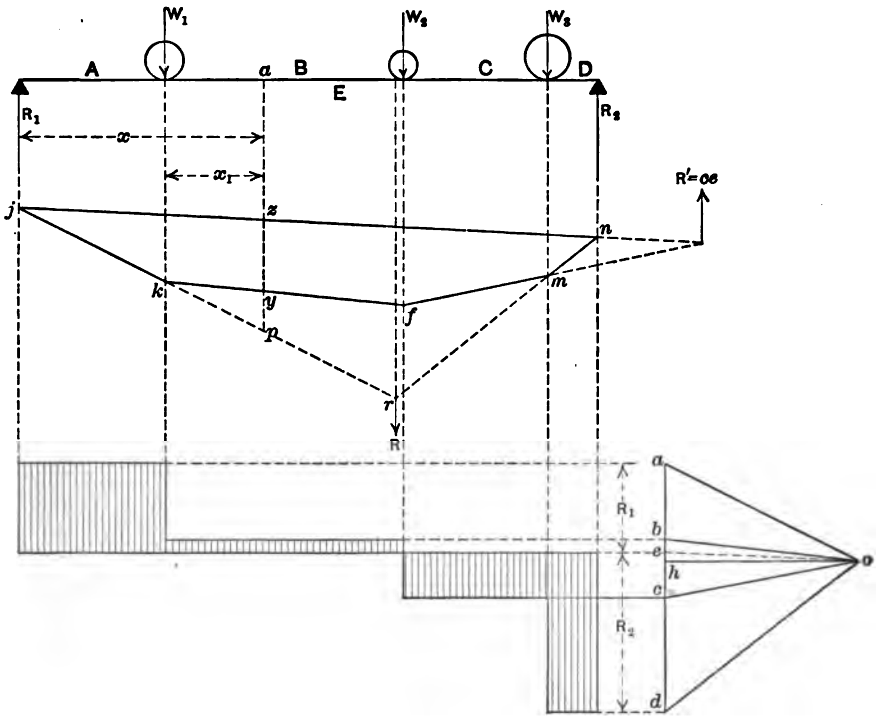


Fig. 84.

To find R_1 and R_2 we must construct the funicular polygon of the forces. Select at random some point *o*, distant *oh* from the load line *ab*, as a pole and draw the vectors *oa*, *ob*, *oc*, and *od*. From some point *j* in the line of action of R_1 draw a line parallel to the vector *oa*, and from its point of intersection, *k*, with the vertical from W_1 draw a line parallel to the vector *ob*, and from its point of intersection, *f*, with the line of action of W_2 draw a line parallel to vector *oc*, and from its point of intersection, *m*,

with the line of action of W_3 draw a line parallel to the vector *od*. This last line intersects the line of action of R_2 at n. Join n with j, and we have the funicular polygon *jkfmn* with the external forces of the beam acting at its joints. This funicular polygon has five sides, while there are but four vectors in the force polygon, and since there must be a vector for each side of the funicular polygon, the vector *oe*, parallel to *nj*, must be drawn. The point e is thus determined, and *de* represents in magnitude and direction the support reaction $DE = R_2$ to the scale adopted; and in like manner *ea* represents in magnitude and direction the support reaction $EA = R_1$. By measurement *de* is $\frac{51}{60}$ inch, which, reduced to scale, equals $\frac{51}{60} \times 150 = 127.5$ pounds $= R_2$; and *ea* measures $\frac{27}{60}$ inch, which, reduced to scale, is 67.5 pounds $= R_1$. These are the same values found before for R_1 and R_2.

It will not be necessary to letter the funicular polygon in order to know the names of the members, as an examination of the force polygon at once discloses them. Thus, the member parallel to *oa* is OA, the one parallel to *ob* is OB, and so on. The names of the members may also be determined from the lettering of the beam, since the length of each member is terminated by the lines of action of two forces of the beam. For example, the member *kf* is terminated by the lines of action of W_1 and W_2, and since the letter B appears between W_1 and W_2 the member *kf* is named OB, or simply B.

The resultant, *ad*, of W_1, W_2, and W_3 acts through r (see Art. 57). Should it be desired to replace two or more of the forces by their resultant, its magnitude and a point in its line of application may be determined. Thus, if it were desired to replace the forces W_3 and R_2 by their equivalent, we find from the force diagram that the resultant of *cd* and *de* is *ce* in magnitude and direction, and a point in the line of application is found by producing OE and OC to their intersection as shown.

59. The Funicular Polygon a Bending-moment Diagram. — Consider any section a of the beam of Fig. 84. Produce jk to its intersection with the vertical from a at p. Draw the horizontals x and x_1 and regard them as the altitudes of the triangles jpz and kpy respectively. All the triangles of the force polygon have the same altitude oh.

The triangles jpz and kpy are respectively similar to the triangles oea and oba, having their sides mutually parallel. Hence we have

$$\frac{x}{oh} = \frac{pz}{ea} = \frac{pz}{R_1}, \quad \text{whence} \quad R_1 x = pz \times oh,$$

and

$$\frac{x_1}{oh} = \frac{py}{ab} = \frac{py}{W_1}, \quad \text{whence} \quad W_1 x_1 = py \times oh.$$

The bending moment at the section a is,

$$M_a = R_1 x - W_1 x_1 = pz \times oh - py \times oh = oh\,(pz - py) = yz \times oh.$$

That is, the bending moment at any section of the beam is equal to the product of the ordinate of the funicular polygon at the section and the polar distance. Thus, the ordinate yz under the section a measures $\frac{29}{80}$ inch, and the polar distance oh measures 1 inch, representing 150 pounds to the scale selected. Then the bending moment at section a is,

$$M_a = \tfrac{29}{80} \times 4 \times 150 = 217.5 \text{ pounds-feet,}$$

as was found on page 33.

The shear diagram of the beam can readily be constructed by projection from the force diagram. Thus, the shear at any section between the left support and W_1 is $R_1 = ea$, and is plotted by projecting ea horizontally as shown. At any section between W_1 and W_2 the shear is $R_1 - W_1 = ea - ab = eb$, and is plotted by projecting eb. The shear at any section between W_2 and W_3 is $R_1 - W_1 - W_2 = ea - ab - bc = - ec$, and between W_3 and the right support the shear is $R_1 - W_1 - W_2 - W_3$

$= ea - ab - bc - cd = - ed = - R_2$. Projecting these two shears, the diagram is completed.

It has been stated that the pole of the force diagram may be selected at random, but if it be selected so that its distance from the load line be some definite number expressed to scale in units of the load, then a bending-moment scale may be obtained which will enable the bending moment to be measured directly from the diagram and obviate the necessity of multiplying each measurement by the polar distance. The pole, o, of the force diagram of Fig. 84 was taken at a distance of 1 inch from the load line ab, the polar distance oh, therefore, representing 150 pounds. Then, since the linear scale is 1 inch = 4 feet, an ordinate measuring 1 inch represents 4 feet; but as this must be multiplied by the polar distance we shall have:

$$1 \text{ inch} = 4 \text{ feet} \times 150 \text{ pounds} = 600 \text{ pounds-feet},$$

or $$\tfrac{1}{60} \text{ inch} = 10 \text{ pounds-feet},$$

a new and convenient scale by which the bending moments can be measured directly from the diagram. The bending-moment scale is derived in each instance by multiplying the linear scale by the polar distance expressed in pounds or tons. The ordinate yz of the funicular polygon, Fig. 84, measures 21.75 sixtieths of an inch, and the bending moment at the section a of the beam is, therefore, $21.75 \times 10 = 217.5$ pounds-feet.

A few examples of the application of the funicular polygon to beams will be given.

Example I. — A beam supported at the ends is 20 feet long, weighs 400 pounds, and has concentrated loads of 360 pounds and 440 pounds at 8 feet from the left end and 4 feet from the right end respectively. Draw the bending-moment and shear diagrams, and measure the bending moments under the concentrated loads and the shear stress at the middle of the beam.

Solution. — Select a linear scale of $\frac{1}{8}$ inch = 1 foot, and a load scale of 1 inch = 400 pounds.

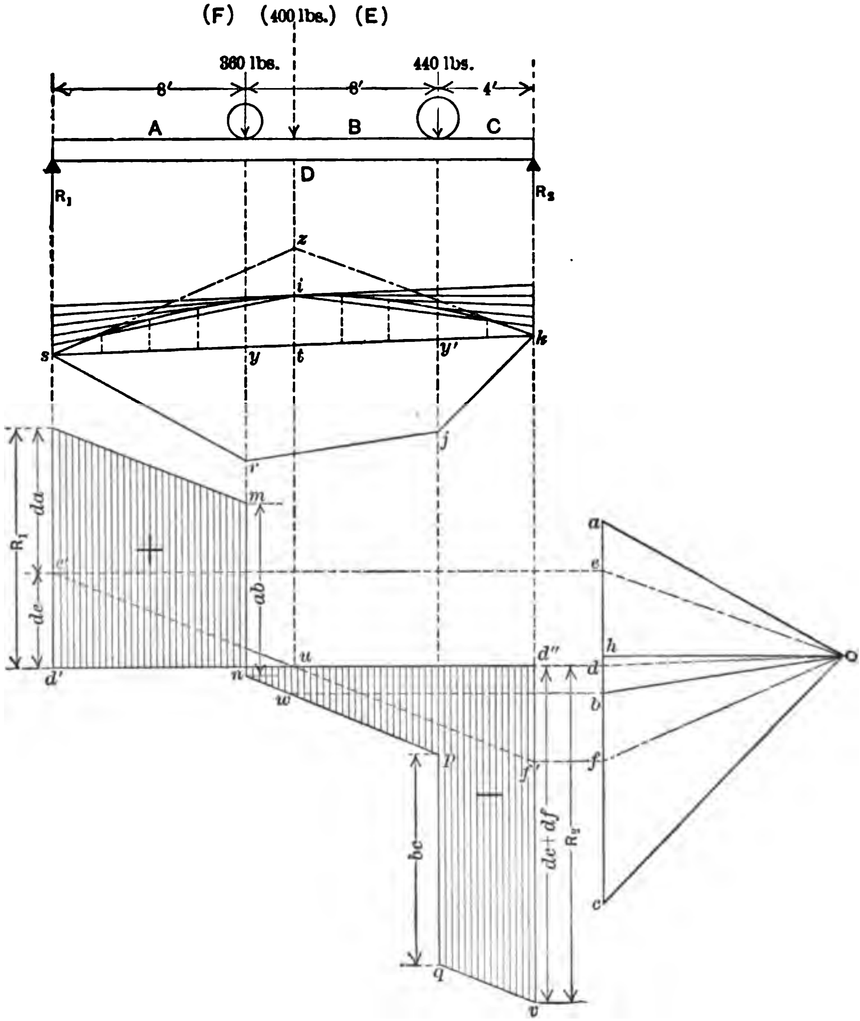
Set out the beam as shown in Fig. 85, and since the beam weighs 400 pounds it has, in addition to the concentrated loads, a uniformly distributed load of 20 pounds per foot.

We shall first construct the funicular polygon for the concentrated loads, neglecting for the present the uniformly distributed load.

Set off the load line *ac* by making *ab* measure 0.9 inch and *bc* measure 1.1 inches to represent the loads of 360 pounds and 440 pounds respectively. Select the pole *o* at a distance of 1.25 inches from the load line, so that the polar distance will represent $1.25 \times 400 = 500$ pounds. We shall then have for the bending-moment scale, $\frac{1}{8}$ inch = 1 foot $\times$ 500 pounds = 500 pounds-feet, or $\frac{1}{40}$ inch = 100 pounds-feet.

Draw the vectors, *oa*, *ob*, and *oc*. From a point *s* in the line of action of R_1 draw a parallel to *oa* and produce it until it intersects the line of action of W_1 at the point *r*. From *r* draw a parallel to *ob*, producing it until it intersects the line of action of W_2 at the point *j*. From *j* draw a line parallel to *oc* and produce it until it intersects the line of action of R_2 at the point *k*. Join *k* with *s*. Then *js* is the closing line of the funicular polygon *srjk*. Draw *od* parallel to *sk*. Then *cd* and *da* represent in magnitude and direction the support reactions R_2 and R_1 respectively, and the funicular polygon *srjk* is the diagram of bending moments for the concentrated loads.

The bending-moment diagram of the evenly distributed load of 400 pounds will be parabolic in form, and it will be convenient to construct its funicular on the closing line *ks* of the funicular of the concentrated loads. To do so we will take the same pole, *o*, as was used for the concentrated loads, and will consider the whole of the distributed load of 400 pounds to be concentrated at the middle of the beam, as shown in the figure. This will add


(F) (400 lbs.) (E)
360 lbs.
440 lbs.
8′
8′
4′
A
B
C
D
R_1
R_2
z
i
s
y
t
y′
k
r
j
m
+
−
R_1
da
de
ab
bc
de+df
R_2
e′
d′
n
u
w
d″
p
f′
q
v
a
e
h
d
b
f
c
o

. Fig. 85.

200 pounds each to R_2 and R_1. On each side of d in the load line lay off ed and df, each one-half inch in length, to represent these additions to the support reactions. Draw the vectors oe and of. From k and s draw parallels to oe and of respectively. They intersect at z. Then zsk is the funicular polygon for the distributed load, supposing it to be concentrated at the middle of the beam. It has been shown in Art. 18 that the bending moment at the middle of a simple beam uniformly loaded is only one-half that due to the same load concentrated at the middle. Hence, the ordinate ti measures the bending moment at the middle of the beam due to the distributed load, i being the middle point of tz. A parabola constructed on sk as a chord, and passing through the point i, is the bending-moment diagram due to the distributed load, and $srjki$ is the complete bending-moment diagram for the beam.

The reaction R_2 measured on the load line is $cd + df = \frac{87}{50}$ inches; hence, $R_2 = \frac{87}{50} \times 400 = 696$ pounds. The reaction $R_1 = da + de = \frac{63}{50}$ inches; hence, $R_1 = \frac{63}{50} \times 400 = 504$ pounds.

The ordinates y and y' under W_1 and W_2 measure $\frac{34}{40}$ inch and $\frac{26}{40}$ inch respectively. The bending moments under W_1 and W_2 are therefore 3400 pounds-feet and 2600 pounds-feet respectively. These results may be checked easily by calculation.

Commencing at the left support the shear due to the uniform load is equal to half that load, and gradually decreases from left to right until, at the middle, it becomes zero, and at the right support it becomes $\frac{wL}{2} - wL = -\frac{wL}{2}$, or to one-half the load, but negative, w denoting the load per unit of length. The diagram $d'e'f'd''$, therefore, represents the shear due to the distributed load.

The total shear at the left support is R_1 and is equal to $da + de$. The parallel to $e'f'$ shows the gradual decrease of the shear from the left support to W_1 due to the uniform load. Passing W_1

the shear suddenly drops to n and becomes negative, mn being equal to ab. The parallel to $e'f'$ drawn from n shows the gradual negative increase in the shear from W_1 to W_2 due to the distributed load. Passing W_2 the shear suddenly drops to q, pq being equal to bc. The parallel to $e'f'$ drawn from q shows the further increase in the shear due to the distributed load until, at v, it becomes $-R_2 = -(dc + df)$. The ordinate uw at the middle of the beam measures $\frac{7}{50}$ inch; the shear at the middle section is, therefore, $\frac{7}{50} \times 400 = 56$ pounds.

Example II. — The beam with overhanging ends of Fig. 52, p. 52, loaded uniformly with 20 pounds per foot and with two concentrated loads, may be solved readily by means of the funicular polygon. The concentrated loads W_1 and W_2 are 200 pounds and 400 pounds respectively.

Set off the beam as shown in Fig. 86. Divide the beam into any convenient number of parts, eight in this instance, so that each part will be 4 feet in length and will bear 80 pounds of the distributed load. These eight parts constitute as many external forces, each acting at its center of gravity.

Letter the beam according to the Bow system, and adopt the following scales: Linear, 0.1 inch = 1 foot; load, 1 inch = 400 pounds. We shall then have: $W_1 = \frac{200}{400} = 0.5$ inch; $W_2 = \frac{400}{400} = 1$ inch; and the load of each of the equal divisions will be represented by $\frac{80}{400} = 0.2$ inch.

Set off the load line ak accordingly. Select a pole o distant 1.5 inches from the load line, so that the polar distance will represent 600 pounds. We shall then have: $\frac{1}{10}$ inch = 1 foot $\times$ 600 = 600 pounds-feet, or $\frac{1}{60}$ inch = 100 pounds-feet, for the bending-moment scale.

Draw the vectors of the force polygon. From some point m in the line of action of R_1 draw a parallel to oa and produce it until it intersects the line of action of AB at n. Through n draw a parallel to ob, producing it to its intersection p with

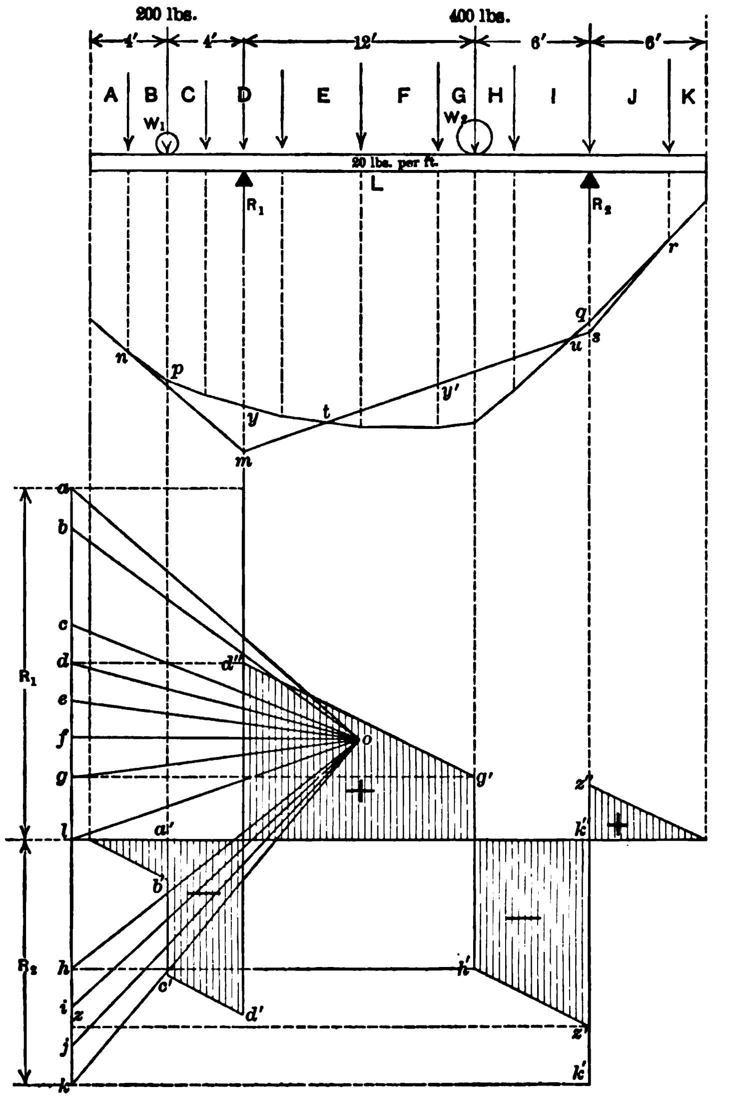

200 lbs.
400 lbs.
4'
4'
12'
6'
6'
A B C D E F G H I J K
W_1
W_2
20 lbs. per ft.
L
R_1
R_2
n p y t m y' q u s r
a b c d e f g l h i j k
d'' o g' a' k' z' b' c' d' h' z
+
−

Fig. 86.

the line of action of BC, or of W_1. Draw parallels to the other vectors as shown. The member qr, parallel to oj, intersects the line of action of JK at r. The parallel to ok through r intersects the line of action of R_2 at s. The closing member is, then, sm, and mnp . . . qrs is the funicular polygon, or the bending-moment diagram of the beam.

Draw ol parallel to sm. Then kl and la represent to scale the reactions R_2 and R_1 respectively.

It should be noted that ordinates within $mnpt$ and sur give negative bending moments, and that t and u are the points of inflection. The ordinates y and y' give the maximum negative and positive bending moments respectively, and as they measure $\frac{14.4}{60}$ and $\frac{16}{60}$ of an inch respectively, the bending moments are 1440 pounds-feet and 1600 pounds-feet.

Commencing at the left end of the beam, the shear increases from zero to $-ab$ when W_1 is reached. Passing W_1 the shear becomes $-ab - bc = -ac$, and increases up to the left support, where it becomes $-(ab + bc + cd) = -ad$. Passing the left support it becomes $R_1 - ad = ld$ and decreases up to W_2, where it becomes $ld - dg = lg$. Passing W_2 it becomes $lg - gh = -lh$, and gradually increases until the right support is reached, where it becomes $-lh - hz = -lz$ (the center of gravity of the weight on the seventh division of the beam happening to fall directly over the right support, one-half of that weight (IJ) actually lies to the left and one-half to the right of the support, and must be so considered). Passing the right support the shear again becomes positive and equal to $R_2 - lz = zk$, and then decreases to zero at the right end of the beam.

Example III. — To draw the bending-moment and shear diagrams of the cantilever with concentrated loads, as shown in Fig. 87, we proceed as follows:

Draw the load line ad, making ab, bc and cd equal, to some

selected scale, to the loads *AB*, *BC*, and *CD* respectively. Select some pole *o*, and draw the vectors *oa*, *ob*, *oc*, and *od*. From some point *r* in the support line draw a parallel to *oa*, producing it until it intersects the line of action of *AB* at *q*. From *q* draw a parallel to *ob* and produce it until it intersects the line of action of *BC* at *p*. From *p* draw a parallel to *oc* and produce it until it intersects the line of action of *CD* at *n*. From *n* draw a parallel to *od* to meet the line of support at *m*. Then *mnpqr* is

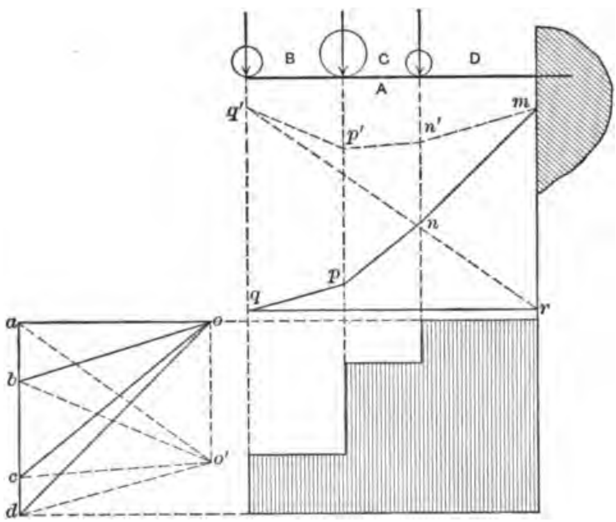


Fig. 87.

the bending-moment diagram of the cantilever, and the ordinate under any section of the beam is the measure of the bending moment at that section. The maximum bending moment is, of course, at the wall.

The shear diagram is drawn by projection from the load line and presents no difficulties.

In the case of a cantilever it is found convenient to select the pole at some chosen perpendicular distance from either extrem-

ity of the load line. In Fig. 87 the pole o might just as well have been chosen at o' and an equal bending-moment diagram, $mn'p'q'r'$, constructed, as shown by the dotted lines.

Example IV. — The cantilever of Fig. 39, p. 40, having a combination of concentrated and uniformly distributed loads may be solved easily by means of the funicular polygon.

Using the same linear scale of $\frac{2}{10}$ inch = 1 foot as for Fig. 39, and a load scale of 400 pounds to the inch, we shall first construct the funicular polygon for the concentrated loads.

Reduced to scale, $AB = \frac{32}{400} = \frac{8}{100}$ inch, and $BC = \frac{44}{400} = \frac{11}{100}$ inch. Set off the load line ac, Fig. 88, making ab and bc equal to $\frac{8}{100}$ inch and $\frac{11}{100}$ inch respectively.

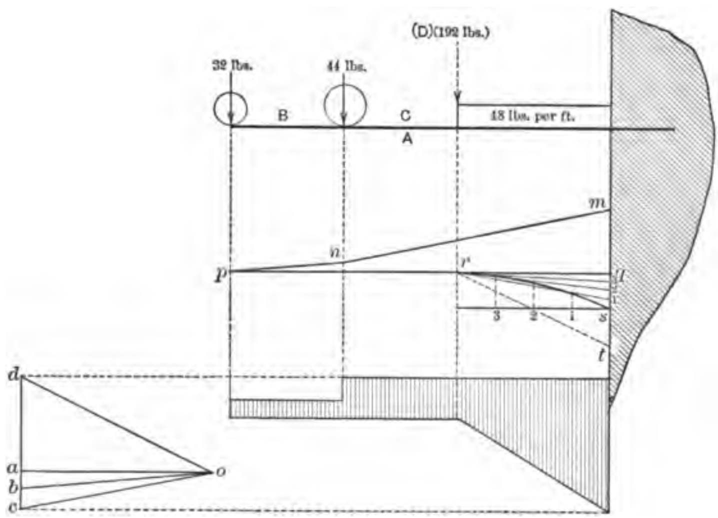


Fig. 88.

Choose a pole o at a perpendicular distance of 1 inch from a, so that the polar distance oa represents 400 pounds. We shall then have for the bending-moment scale, $\frac{2}{10}$ inch = 1 foot × 400 pounds = 400 pounds-feet, whence $\frac{1}{20}$ inch = 40 pounds-feet. Draw the vectors oa, ob, and oc.

Commencing at some point q in the wall line, draw qp parallel to oa, pn parallel to ob, and nm parallel to oc, thus forming the funicular polygon, or bending-moment diagram, $mnpq$ for the concentrated loads.

The cantilever is uniformly loaded with 48 pounds per foot for a distance of 4 feet from the wall, making a total uniform load of 192 pounds. Assuming this load concentrated at the outer extremity of the 4 feet, set off ad equal to $\frac{192}{400} = \frac{24}{50}$ inch to represent it (the uniform load AD is taken contraclockwise in order to join its funicular to the line qp of the funicular polygon of the concentrated loads). Draw the vector od, and from r, the intersection of pq with the vertical at 4 feet from the wall, draw rt parallel to od. It can easily be shown that the bending moment due to a concentrated load at the end of a cantilever is twice that due to the same load uniformly distributed, so the distance qt must be bisected at s, and the parabola having its apex at r, and passing through s, gives rsq as the bending-moment diagram due to the distributed load. Then $mnprs$ is the complete bending-moment diagram of the cantilever. The ordinate ms at the wall measures $\frac{25.3}{50}$ inch, and the bending moment is, therefore, $25.3 \times 40 = 1012$ pounds-feet, the same as found on page 41.

The shear diagram is drawn by projection from the load line and presents no difficulties.

Example V. — A circular steel axle, supported on end journals, is 10 feet long between journal centers and subjected to a load of 12,000 pounds at a point 4 feet from the center of the journal at the right end. Determine the dimensions of the axle.

Solution. — The axle is to be regarded as being subjected only to a bending stress.

To a scale of $\frac{3}{8}$ inch = 1 foot, lay off the axle in skeleton form,

Fig. 89. To a load scale of $\frac{1}{8}$ inch = 1200 pounds, lay off the load line *ab*, 1.25 inches in length, to represent the load of 12,000 pounds. Select a pole *o* one inch from *ab*, so that the polar

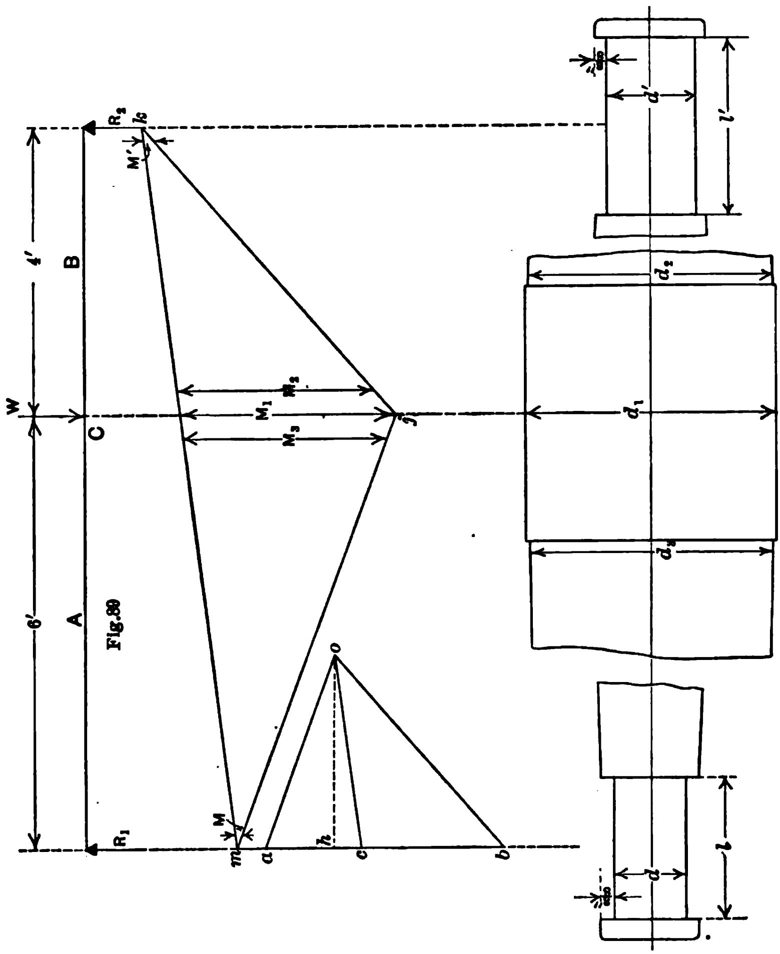


Fig. 89.

Fig. 90.

distance *oh* represents 9600 pounds. Then, $\frac{3}{8}$ inch = 1 foot × 9600 pounds = 9600 pounds-feet, or $\frac{1}{40}$ inch = 640 pounds-feet, a convenient bending-moment scale.

Draw the vectors *oa* and *ob*. Commencing at a point *m* in the line of action of R_1, draw the funicular polygon *mjk*. Draw *oc* parallel to the closing line *km*. Then *bc* and *ca* are the reactions R_2 and R_1 respectively, in magnitude and direction. By scale measurement $bc = 7200$ pounds $= R_2$, and *ca* measures 4800 pounds $= R_1$.

Treating the journals as uniformly loaded cantilevers, we have $\frac{R_1 l}{2}$ as the maximum bending moment of the journal at the left end, in which l denotes the length of the journal.

The resisting moment is $\frac{SI}{c} = \frac{S\pi d^4}{64\,c} = \frac{S\pi d^3}{32} = 0.196\,Sd^3$, in which d denotes the diameter of the journal. From this it is seen that the resisting moment, and therefore the bending moment, is proportional to the cube of the diameter.

Then, $$\frac{R_1 l}{2} = 0.196\,Sd^3, \quad \text{whence} \quad d = \sqrt{\frac{R_1}{0.196\,S} \cdot \frac{l}{d}}.$$

It is usual to fix the ratio $\frac{l}{d}$ in accordance with the conditions of the case. Assuming the direction of the load to be constant and the maximum revolutions to be 250 per minute, it is good practice to make $\frac{l}{d} = 2$; for slower speeds the ratio would be less. Then, taking S at 10,000 pounds per square inch, we have

$$d = 0.0226\sqrt{4800 \times 2} = 2.21 \text{ inches, say } 2\tfrac{1}{4} \text{ inches;}$$

whence $$l = 4\tfrac{1}{2} \text{ inches.}$$

For the journal at the right end,

$$d' = 0.0226\sqrt{7200 \times 2} = 2.71 \text{ inches, say } 2\tfrac{3}{4} \text{ inches; and}$$
$$l' = 5.5 \text{ inches.}$$

Since the bending moment at any section of the axle is proportional to the cube of the diameter at the section, it follows that the diameter at any section is proportional to the cube root

of the bending moment at the section. The lengths l and l' being known, the bending moments M and M' may be measured from the diagram or calculated from $\frac{R_1 l}{2}$ and $\frac{R_2 l'}{2}$ respectively. These moments are found to be: $M = 625$ pounds-feet, and $M' = 1200$ pounds-feet. The bending moment M_1 at the load measures 28,800 pounds-feet.

Then, $$\frac{d_1}{d'} = \frac{d_1}{2.71} = \sqrt[3]{\frac{M_1}{M'}} = \sqrt[3]{\frac{28,800}{1200}} = 2.88,$$

whence $d_1 = 7.8$ inches, say $7\frac{13}{16}$ inches.

Making the hub seat of the wheel 8 inches long, we have now to determine d_2 and d_3. The bending moments M_2 and M_3 measure 26,240 pounds-feet and 27,200 pounds-feet respectively.

Then, $$\frac{d_2}{d'} = \frac{d_2}{2.71} = \sqrt[3]{\frac{26,240}{1200}} = 2.796,$$

whence $d_2 = 7.577$ inches, say $7\frac{5}{8}$ inches.

and $$\frac{d_3}{d} = \frac{d_3}{2.21} = \sqrt[3]{\frac{27,200}{625}} = 3.517,$$

whence $d_3 = 7.773$ inches, say $7\frac{3}{4}$ inches.

An empirical rule in machine design makes the height of the collar of the journal $\frac{d}{10} + \frac{1}{8}$ inch, and its width 1.5 times the height. Hence,

Height of journal collar at left end $= \frac{2.21}{10} + \frac{1}{8} = \frac{11}{32}$ inches,

whence width $= \frac{17}{32}$ inches.

Height of journal collar at right end $= \frac{2.75}{10} + \frac{1}{8} = \frac{3}{8}$ inches,

whence width $= \frac{9}{16}$ inch.

The axle is drawn in Fig. 90 to the scale of one-sixth.

Example VI. — The total weight on the 4 axles of a semiconvertible street car is 46,560 pounds. Motion is conveyed to

the axles from the motors, one for each axle, by means of gears, and the weight is carried on journals that overhang the car-wheel journals 9.25 inches. The horse power of each motor is 40, and the maximum speed on level is 25 miles per hour. The diameter of the car wheels is 33 inches and the gauge of the road 5 feet. Determine the dimensions of the axles.

Solution. — A car axle is subjected to bending only when the car is moving on a straight portion of the road, but in rounding curves it is subjected to both torsion and bending, so that the equivalent twisting moment must form the basis for the determination of the dimensions of the axle.

The load carried on each journal is $\frac{W}{8} = \frac{46,560}{8} = 5820$ pounds, which occasions an equal reaction at the journal of each car wheel. The condition then is one of a beam having equal overhanging ends, each overhang being uniformly loaded with $\frac{W}{8}$ or, which is the same thing, having $\frac{W}{8}$ concentrated at its middle.

To a linear scale of $\frac{3}{5}$ inch = 1 foot, or of one-twentieth, lay off the axle in skeleton form, Fig. 91. To a load scale of $\frac{1}{8}$ inch = 582 pounds lay off *ab* 1.25 inches in length to represent the load *AB* of 5820 pounds, and *bc* the same length to represent the equal load *BC*. Select a pole *o*, distant 1.25 inches from the load line *ac*, so that the polar distance will represent 5820 pounds. Then, $\frac{3}{5}$ inch = 1 foot × 5820 pounds = 5820 pounds-feet, or $\frac{1}{50}$ inch = 194 pounds-feet, which is a convenient bending-moment scale.

Draw the vectors *oa*, *ob*, and *oc*. Commencing at some point *m* in the line of action of R_1, draw *ma* parallel to *oa*, and from *a*, the point of intersection of *ma* with the line of action of *AB*, draw *an* parallel to *ob*, and from *n*, its intersection with the line of action of *BC*, draw *nk* parallel to *oc*, intersecting the line of action of R_2 at *k*. Join *k* with *m*. Then *mank* is the funicular

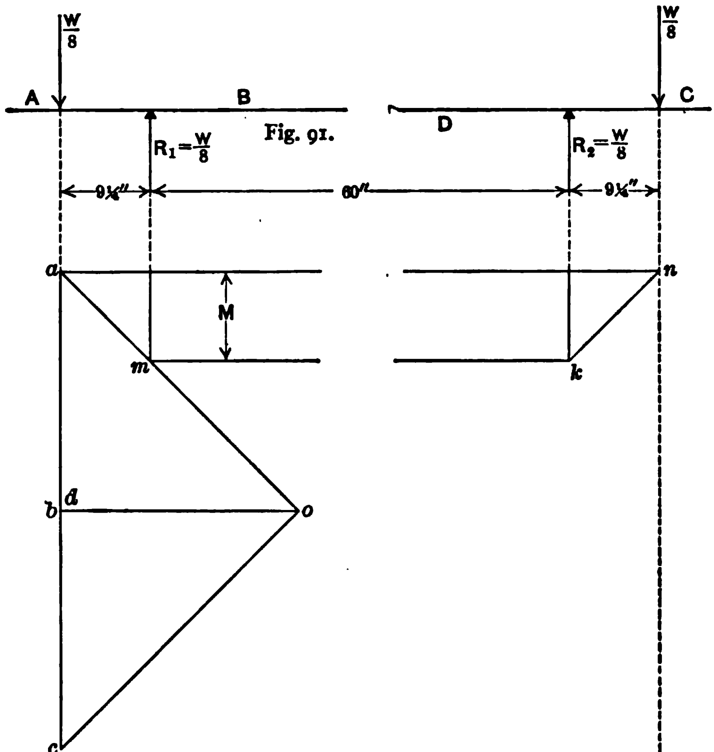


Fig. 91.

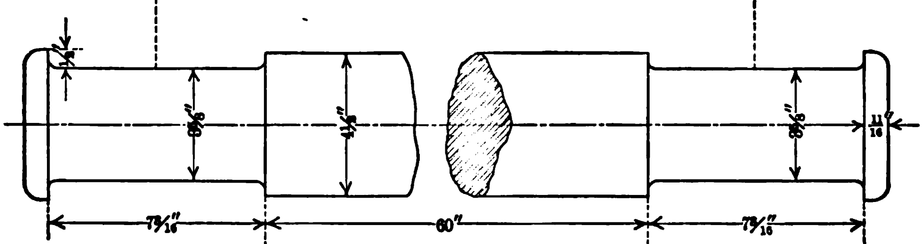


Fig. 92.

polygon, or bending-moment diagram, of the axle. Draw *od* parallel to *km*; it will, of course, coincide with *ob*, since *ab* and *bc* are equal and *o* was taken on the horizontal through *b*.

The bending moment M between the supports is constant and measures $\frac{23.2}{50} \times 194 = 4500$ pounds-feet.

To find the twisting moment we must know the revolutions per minute of the axle.

$$\text{Circumference of car wheel} = \frac{3.1416 \times 33}{12} = 8.64 \text{ feet.}$$

$$\text{Speed of car per minute} = \frac{5280 \times 25}{60} = 2200 \text{ feet.}$$

$$\text{r.p.m.} = \frac{2200}{8.64} = 254.63.$$

$$Pr = M_t = \frac{40 \times 33{,}000}{2 \times 3.1416 \times 254.63} = 825 \text{ pounds-feet.}$$

To find the equivalent twisting moment (see Art. 47) we have

$$M_e = M + \sqrt{M^2 + M_t^2} = 4500 + \sqrt{(4500)^2 + (825)^2}$$
$$= 9075 \text{ pounds-feet} = 108{,}900 \text{ pounds-inches.}$$

Taking S as 6000 (see Art. 46), we shall have

$$108{,}900 = 0.196\,Sd^3 = 0.196 \times 6000\,d^3,$$

in which d is the diameter of the axle.

Then $$d = \sqrt[3]{\frac{108{,}900}{1176}} = 4.52 \text{ inches, say } 4\tfrac{1}{2} \text{ inches.}$$

Treating the overhanging journals as cantilevers uniformly loaded, or with the load $\frac{W}{8}$ concentrated at the middle, the maximum bending moment is $M = 4500$ pounds-feet. For the diameter of the journals we shall then have

$$4500 \times 12 = 0.196\,Sd_1^3, \text{ whence } d_1 = \sqrt[3]{\frac{54{,}000}{1176}} = 3.58 \text{ inches.}$$

Or, since the bending moments are proportional to the cubes of the diameters, we shall have, denoting the diameter of the journals by d_1,

$$\frac{4.52}{d_1} = \sqrt[3]{\frac{9075}{4500}} = 1.26, \text{ whence } d_1 = 3.59 \text{ inches, say } 3\tfrac{5}{8} \text{ inches.}$$

Denoting the length of the journals by l, and taking the ratio $\frac{l}{d_1}$ as 2, we shall have

Length of journal = $3.59 \times 2 = 7.18$ inches, say $7\tfrac{3}{16}$ inches.

The height of collar at outer end of journal $= \frac{3.59}{10} + \frac{1}{8} =$ 0.484 inch, say $\frac{1}{2}$ inch.

Width of collar $= 0.484 \times 1.5 = 0.7$ inch, say $\frac{11}{16}$ inch.

The diameter of the axle has been determined under the supposition that the gear wheel which receives the power from the motor is fitted to the axle by hydraulic pressure. Should the gear wheel be keyed to the axle the diameter would have to be increased by an amount equal to the depth of the keyway, amounting in this instance to about $\frac{5}{16}$ inch.

The axle is drawn in Fig. 92 to the scale of one-sixth.

Example VII. — A steel axle rests on two journals and is subjected to a vertical load of 6000 pounds applied on an overhanging end at a distance of 15 inches from the center of the nearest journal. The distance between the journals is 4 feet. Construct the funicular polygon, and determine the magnitude and direction of the reactions.

Solution. — To a linear scale of $\frac{3}{4}$ inch = 1 foot lay off the axle in skeleton form, Fig. 93. To a load scale of $\frac{1}{4}$ inch = 1500 pounds lay off the load line ab one inch in length to represent the load AB of 6000 pounds. From the pole o, distant 1 inch from ab, draw the vectors oa and ob. The polar distance of 1 inch represents 6000 pounds to scale, and we shall have

$\frac{3}{4}$ inch = 1 ft. $\times$ 6000 lbs. = 6000 lbs.-ft., or $\frac{1}{40}$ inch = 200 lbs.-ft. as a bending-moment scale.

From a point m in the line of action of R_1 draw a parallel to oa, intersecting the line of action of W at j. From j draw a parallel to ob, intersecting the line of action of R_2 at k, and from k draw km as the closing line of the funicular polygon mjk. From o draw a parallel to km, intersecting the load line produced at c. Then bc and ca represent the reactions R_2 and R_1 in magnitude and direction respectively. By scale measurement $bc = R_2 = 7875$ pounds, and $ca = R_1 = 1875$ pounds.

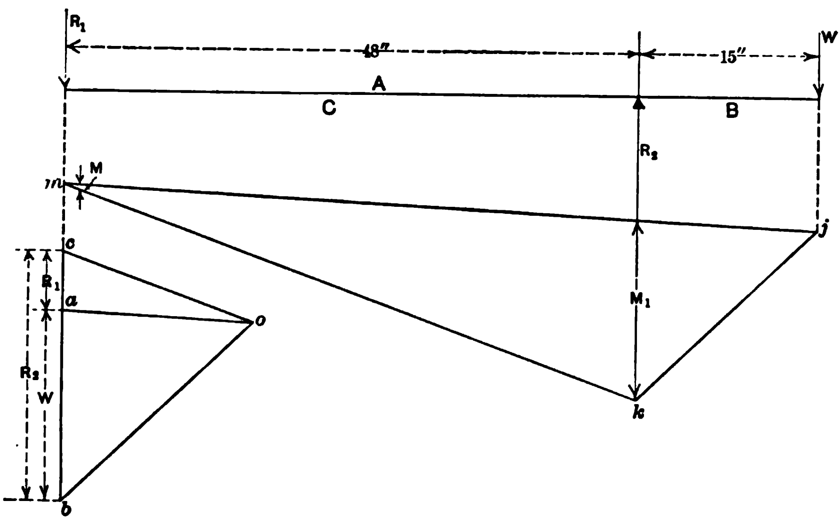


Fig. 93.

PROBLEMS

1. A beam 22 feet long supports a load of 1000 pounds at a point 6 feet from the left end and one of 1200 pounds at 5 feet from the right end. In addition there is a uniformly distributed load of 100 pounds per foot. Construct the bending-moment and shear diagrams. What concentrated load must be placed at the middle of a similar beam in order that the maximum bending moment shall be the same as that of the given beam?

Ans. 2190 lbs.

2. A beam 30 feet long weighs 20 pounds per foot and overhangs each support 6 feet. It bears a superimposed load of 100 pounds per foot, and a load of 1400 pounds concentrated at a point 3 feet to the right of the

middle. Construct the bending-moment and shear diagrams, and find graphically the bending moment at the dangerous section and the distances of the points of inflection from the left support.

Ans. 7760 lbs.-ft.; 1.48 ft. and 16.9 ft.

3. A cantilever 14 feet long supports three concentrated loads,— 500 pounds at 4 feet from the wall, 600 pounds at 10 feet from the wall, and 200 pounds at the extremity. In addition it bears a uniformly distributed load of 80 pounds per foot run. Construct the bending-moment and shear diagrams.

4. Determine the diameters of the journals of the axle of Example VII, page 142.

Ans. $1\frac{3}{4}$ ins. and $4\frac{1}{2}$ ins.

CHAPTER VIII

FRAMED STRUCTURES. RECIPROCAL DIAGRAM

60. Framed Structures. — A framed structure is an assemblage of members for the transmission or modification of external forces, the internal stresses occasioned thereby in the members being principally those of tension and compression. A member in tension is known as a *tie;* if in compression, it is known as a *strut.*

A *frame* is a theoretical structure, the joints connecting its members being supposed frictionless. There are, of course, no such things as frames, since all joints offer some resistance to rotation. In general, however, engineering structures approach so nearly to frames that no sensible error results from treating them as such.

The members of a frame are rigid bars, hinged at the ends, and it is assumed that: (*a*) The pins at the joints are without friction. (*b*) The external forces acting on the frame are applied at the joints.

If the point of application of a load be at some point intermediate between the joints, parallel forces equivalent to the load must be substituted at the joints. If the point of application be midway between joints, the equivalent parallel forces at the joints will each be one-half the load; if the point of application be otherwise than at the middle, then the equivalent parallel forces at the joints may be found by moments.

61. Loads. — The load on a structure consists of the weight of the structure itself and the external forces acting on it. Stationary and moving weights constitute *dead* and *live* loads

respectively. Wind pressure, the weight of the covering of a structure and the weights the structure may support, are external forces. The reactions of the supporting foundations of a structure are external forces, but they are distinguished from the external forces constituting the load by calling them *supporting* forces.

For the equilibrium of a structure the external forces constituting the load must equal the supporting forces, and there must be a balance between the external and the internal forces.

62. Trusses. — Trusses are frames designed to support the roofs of buildings and such loads as are carried by bridges, the supports being widely separated. There are two classes of trusses: Those in which the upper and lower members, called *chord* members, are parallel and horizontal; and those whose chords are not parallel. The members connecting the upper and lower chords are known as *braces*, or as *web members*, and may be vertical or diagonal. The points at which web members meet a chord divide the truss into *bays* or *panels*, and the measurement of a bay is the horizontal distance between its joints.

The triangle being the polygon whose shape cannot be changed without altering the length of its sides, all bridge and roof trusses are made up of triangular frames as the best means of securing rigidity.

Roof trusses are placed from 10 to 16 feet apart, and each truss is known as a *principal*. The upper chord is sometimes known as the *principal rafter*.

63. Distinction between Beams and Girders. — Framed structures, such as are used in bridges and for the support of roofs, are beams in the sense that they are supported at the ends and carry their loads between the supports, but the term *beam* seems to be restricted to the cases where its application is of the simple form and of solid section. Thus, the beam of **I** section, when used alone, is known as a *beam*, but when two of them

are compounded with plates riveted to their top and bottom flanges, the combination is known as a box *girder*, the section of which is not solid. Generally speaking, beams of built-up section are girders.

A truss with its upper and lower chords parallel and horizontal is a direct transformation from the simple I beam, the chord members, like the flanges of the *I*, resisting the bending moment; the web members, like the web of the *I*, resisting the vertical shear and transmitting it from member to member to the supports.

If a chord of a truss is inclined to the horizontal it aids in the transmission of the vertical shear and is not, therefore, designed only to resist the bending moment.

64. Force Action at a Framed Joint. — A structure can remain in a state of rest only when there is equilibrium in the system of forces acting on it, and in order that there shall be a state of rest at a joint there must be equilibrium in the forces acting on it or transmitted to it. Since the joints are to be considered frictionless, the external force acts through the center of the joint, and the action of a member on the pin is balanced by the reaction of the pin on the member, the action and reaction being normal to the surface of contact; and since the joints are in equilibrium, the stresses in the ends of a member must be equal and opposite, therefore the lines of action of the stresses in the members lie in the straight lines joining the centers of the pins.

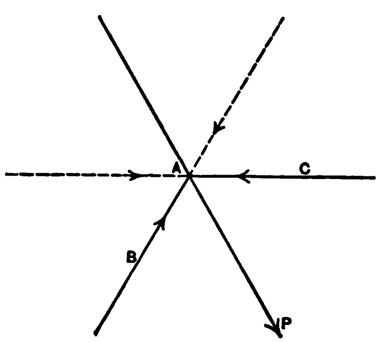


Fig. 94.

If P be an external force acting at the joint A, Fig. 94, its components along the members AB and AC (shown by the dotted lines) are equal and opposite

to the stresses in the members. In all cases the external force at a joint of a frame must be in equilibrium with the stresses in all the members which meet at the joint.

65. Reciprocal or Stress Diagram. — Since there is equilibrium at each joint of a framed structure, it follows that a closed polygon may be constructed whose sides represent the acting forces in magnitude and direction. Such a polygon is known as the *stress diagram* of the frame, but, owing to its interchangeable relations with the frame, it is also known as the *reciprocal diagram* of the frame, and is nothing more nor less than a polygon of forces.

66. Frame Diagram. — As a preliminary to the construction of the reciprocal diagram a scale drawing of the framework must be made, which is known as the *frame diagram*. The complete preparation of the frame diagram comprises the following:

1. The determination of the magnitude and direction of the total load at each joint, replacing any load that may be applied between two joints by equivalent parallel forces at the joints. These equivalent parallel forces are determined in the same manner as that employed in determining the support reactions of a loaded simple beam.

2. The determination of the supporting forces, or support reactions. The reactions can be, and often are, determined from the funicular polygon, but their predetermination affords a check as to the accuracy, and not infrequently furnishes the known external force at a joint where but two members meet, thus providing a starting point for the construction of the reciprocal diagram.

3. The correct lettering of the diagram according to the Bow system.

4. The marking with arrowheads of all the external forces, giving to each its value, and taking care that the arrowheads do not cross the lines of the frame diagram.

67. Support Reactions. — With a small span the ends of a roof truss are *fixed*, and the reactions at the supports are vertical when the loads are vertical. In case of a wind load, acting only on one side of the roof, the reactions due to it act in directions parallel to the normal wind pressure.

With large spans one end of the truss is fixed while the other end is on rollers, thus permitting a lateral movement in case of expansion. The reaction at the fixed end will be inclined and that at the *free* end will be vertical.

It should be noted that when all the external forces are vertical the support reactions are vertical. In such cases the loads that come directly over the supports are omitted, as they have no influence on the stresses in the members. Such loads must be deducted from the total support reactions in order to obtain the proper reactions to use in the determination of the stresses.

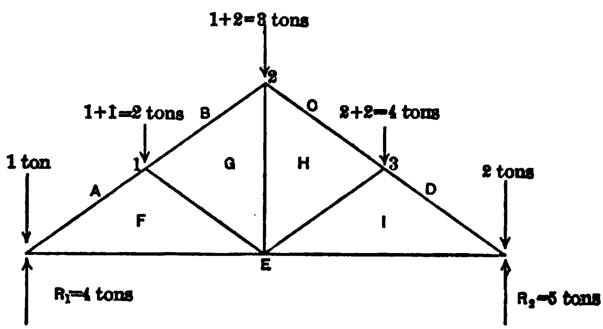


Fig. 95.

For example, Fig. 95 represents a king-post truss with a uniform load of 8 tons on one side and 4 tons on the other. The loads are applied to the joints as follows:

Of the 4 tons on the left-hand rafter we may assume one-half of it borne by the member AF and the other half by the member BG. Distributing these two loads, giving half of each to the ends of the member which supports it, we get a total load of

2 tons at joint 1, 1 ton at the apex end of the rafter, and 1 ton over the left support. Proceeding in a similar manner with the right-hand rafter, we get a total of 3 tons at the apex of the rafters, 4 tons at the joint 3, and 2 tons over the right support.

To find the support reactions we reject the loads over the supports, and, assuming the span to be a, take moments thus:

$$R_1 \times a = 2 \times \frac{3a}{4} + 3 \times \frac{a}{2} + 4 \times \frac{a}{4}, \text{ whence } R_1 = 4 \text{ tons.}$$

$$R_2 \times a = 4 \times \frac{3a}{4} + 3 \times \frac{a}{2} + 2 \times \frac{a}{4}, \text{ whence } R_2 = 5 \text{ tons.}$$

These values of R_1 and R_2 are to be used in determining the stresses in the members, but the total wall reactions are 5 tons and 7 tons for R_1 and R_2 respectively.

68. Wind Pressure. — For purposes of computation the direction of the wind is assumed to be horizontal, and its intensity may be taken as 40 pounds per square foot.

Roofs generally present an inclined surface to the wind, but as the roof structure itself must resist the normal pressure to which it is subjected, it is important to know this normal pressure for the different degrees of roof inclination.

The normal pressures due to a horizontal intensity of 40 pounds per square foot on a vertical surface have been determined experimentally for roofs of different inclinations, and are set forth in the table which follows:

Pitch of roof in degrees.	Normal pressure in pounds per square foot.
5	5
10	10
15	14
20	18
25	22
30	26
35	30
40	33
45	36
50	38
55	39
60	40

For horizontal pressures other than 40 pounds the normal pressures are directly proportional to those given in the table.

69. Drawing the Reciprocal Diagram. — In drawing the reciprocal diagram we proceed as follows:

1. Select a scale — as large as practicable — and draw the force diagram of the external forces, marking the beginning and ending of a line representing a force with the small letters of the alphabet corresponding to the capital letters, taken in clockwise order, found in the spaces flanking the force in the frame diagram. This diagram represents the magnitudes and directions of the external loads on the joints and of the support reactions, and, if properly drawn, forms a closed polygon. In the most frequent case, that of vertical loads, the polygon resolves itself into a straight line, vertical in direction, as already explained. Determine the magnitude of the support reactions, either by moments or by the method of the funicular polygon.

2. Choose as a starting point a joint where a sufficient number of conditions are known to enable its reciprocal to be drawn, being careful to letter the lines of the reciprocal with the small letters of the alphabet corresponding to the capital letters denoting the members in the frame diagram. The lengths of the lines of this reciprocal, to the chosen scale, give the magnitudes of the stresses in the members to which they refer, and the directions, or the kinds, of these stresses are at once determined by noting the directions in which the successive lines of the reciprocal were drawn. The directions of the stresses thus found are at once indicated by placing arrowheads on the members in the frame diagram. Since there are equal and opposite stresses in the ends of a member, arrowheads must now be placed on the other ends of the members whose stresses have been found, making them point in the opposite direction to those placed at the joint whose reciprocal has been drawn.

3. Proceed by similar processes until the reciprocals of all the joints are drawn and the stresses in all the members determined.

It should be noted that the reciprocal of a joint cannot be drawn if the stresses in more than two of the members forming it are unknown. In general terms, the drawing of the reciprocal of a joint of n forces involves $2n$ conditions, viz., n magnitudes and n directions, and unless $2n - 2$ of these conditions are known the data is insufficient to draw the reciprocal.

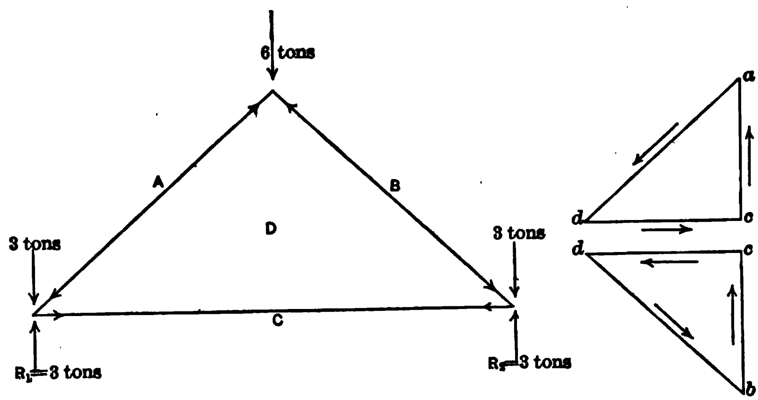


Fig. 96.

Example I. — Suppose each rafter of the roof truss of Fig. 96 to support an evenly distributed load of 6 tons; it is required to find the magnitudes and kinds of stresses in the members of the truss.

We commence by apportioning the loads to the joints of the frame diagram. In doing so, we find a total load of 6 tons at the ridge and a load of 3 tons directly over each supporting wall. As these latter do not affect the stresses in the members they will be omitted from any further discussion, remembering that if the total reactions at the walls are required, R_1 and R_2 must each be increased by 3 tons.

The truss being symmetrical, the support reactions R_1 and R_2 are each equal to half of the load of 6 tons at the ridge.

Letter the frame diagram as shown, ignoring the vertical loads over the supports, so that R_2 will be known as BC and R_1 as CA. Adopt a load scale of 0.25 inch to the ton.

Since each of the three joints of the frame has but two members and a known external force, either may be used as a starting point. Selecting the joint at the left support, we draw *ca* equal in length to 0.75 inch to represent the left reaction CA of 3 tons, and we measure it upward from *c* to *a* because R_1 acts upward. From *a* and *c* draw lines parallel to the members AD and DC respectively. They intersect at *d*, giving the triangle *cad* as the triangle of forces, or the reciprocal diagram, for the joint at the left support. Hence, *ad* and *dc* represent in magnitude and direction the stresses in the members AD and DC. Knowing the direction of R_1, as represented by *ca*, the directions of the actions of the stresses in the two members are known by taking the sides of the triangle in order, as shown by the arrows. The stress in AD acts in the direction *ad*, and that in DC in the direction *dc*. Indicate these directions by placing arrowheads on the two members at points close to the joint. Knowing that the stress in one end of a member is opposed by an equal and opposite stress in the other end, we may now place arrowheads at the other ends of the members AD and DC accordingly, as shown.

To draw the reciprocal of the joint at the right support we draw *bc* upward, and make it 0.75 inch in length to represent the 3 tons of the right reaction BC in magnitude and direction. From *c* and *b* draw lines parallel to the members CD and DB respectively. They intersect at *d*, giving the triangle *bcd* as the reciprocal of the joint at the right support. Hence, *cd* and *db* represent in magnitude and direction the stresses in the members CD and DB. The arrows within the triangle *bcd* show the

direction of these stresses, and they are marked with arrowheads accordingly in the frame diagram. By scale measurements of the reciprocals the stresses in the rafters are found to be 4.4 tons each, and in the tie *CD* the stress is found to be 3.2 tons.

If to the frame of Fig. 96 we add a vertical rod, called the king-post, we obtain the frame of Fig. 97.

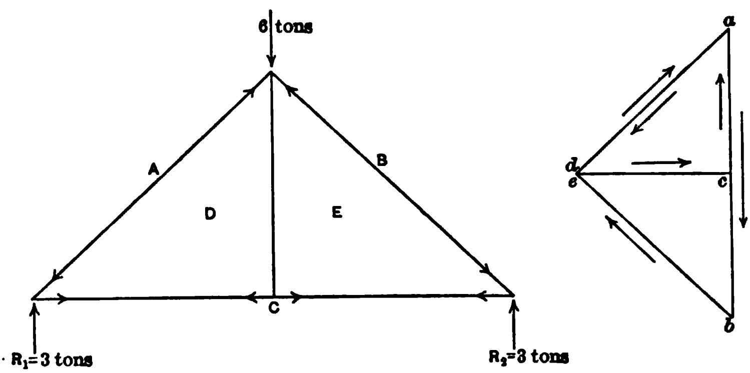


Fig. 97.

To draw the reciprocal of the joint *ABED* at the ridge we measure *ab* downward, and make it 1.5 inches long to represent, to the chosen scale of 0.25 inch to the ton, the magnitude and direction of the external force of 6 tons. The stress in the rafter *AD* has been found to be 4.4 tons, so we draw *ad* parallel to *AD* and make it 1.1 inches long to represent the 4.4 tons to scale. From *b* we draw a line parallel to *BE* and we find it intersects *ad* at *d*. This shows that *d* and *e* are one and the same point, and that there is no stress in the vertical post *DE*. The purpose of *DE* is to prevent sagging in the horizontal member.

As a further illustration we will consider the king-post truss of Fig. 95, reproduced in Fig. 98. The loads and support reactions were found to be as shown.

Knowing the force *EA*, or R_1, we commence at the joint at

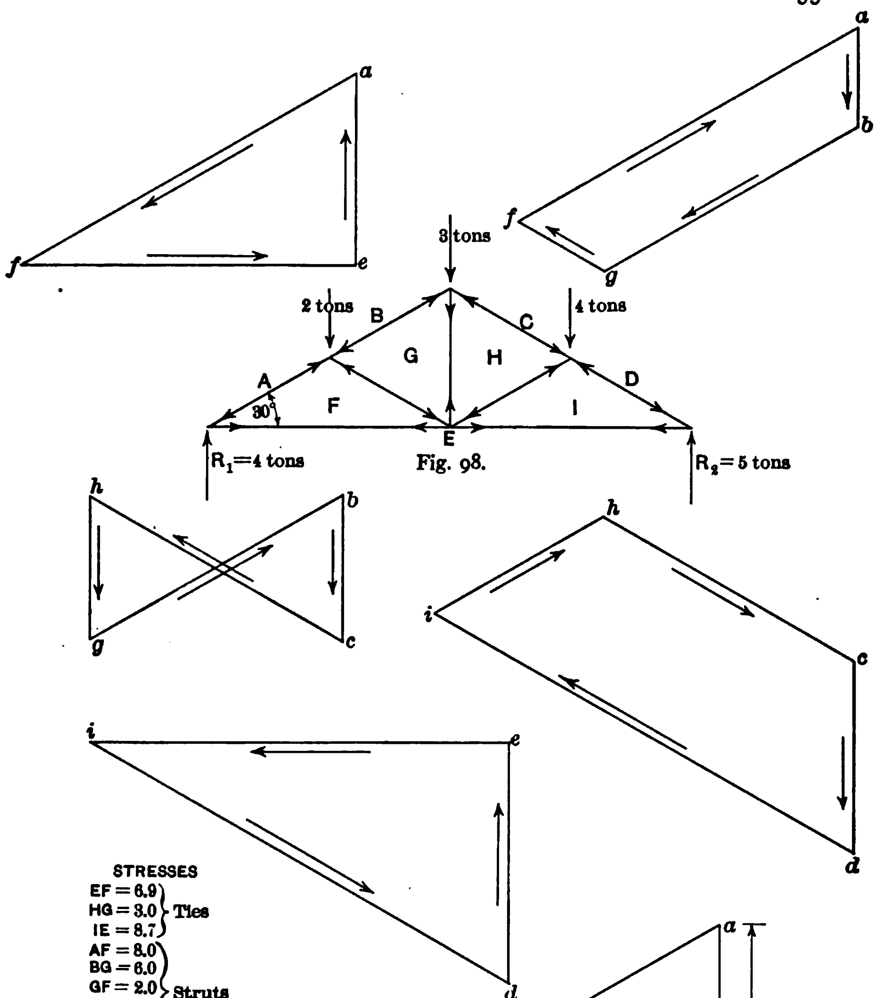

a
f
e
a
b
f
g
3 tons
2 tons
4 tons
B
C
G
H
A
D
F
I
30°
E
R_1=4 tons
R_2= 5 tons
h
b
g
c
h
i
c
d
i
e
d
STRESSES
EF = 6.9
HG = 3.0
IE = 8.7
Ties
AF = 8.0
BG = 6.0
GF = 2.0
CH = 6.0
DI = 10.0
IH = 4.0
Struts

Fig. 98.

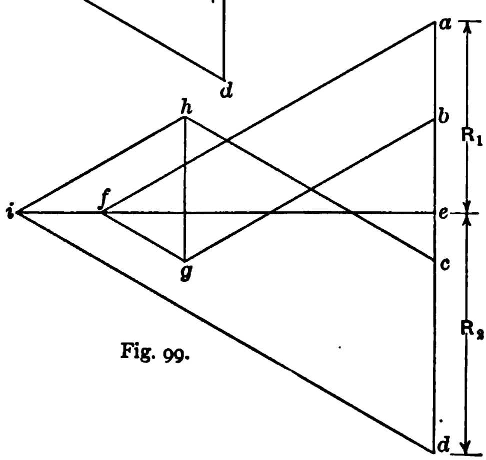

a
h
b
R_1
i
f
e
c
g
R_2
d

Fig. 99.

the left support and, with a scale 0.25 inch to the ton, obtain *eaf* as the triangle of forces for the joint. Marking with arrowheads the kinds of stresses in the members *AF* and *FE*, we proceed to the next joint in clockwise order, that at which the load of 2 tons is applied.

Of the eight conditions of this joint the four directions and two of the magnitudes are known, the magnitude of *FA* having just been determined. We then readily obtain *abgf* as the force polygon of the joint. Marking with arrowheads the kinds of stresses in the members *BG* and *GF*, we proceed in clockwise order to the other joints, obtaining for the joint at the ridge the force polygon *gbch*; for the joint *HCDI* the polygon *hcdi*; and for the joint at the right support the triangle *ide*, thus completing the determination of the stresses in all the members.

It will be observed that each of the force polygons of the joints, when taken in order, contains a side of the one immediately preceding it; hence, each polygon can be built upon the one preceding it, and so produce one figure which will contain all the sides and be a graphic representation of the magnitudes and directions of all the external forces and internal stresses of the structure. Such a figure is the reciprocal diagram of the structure.

The reciprocal diagram of the truss of Fig. 98 is shown in Fig. 99, the load line *ad* having first been drawn and the lines representing the stresses then drawn in their regular order. The magnitudes of the stresses were measured to the scale and found to be as shown in the table.

Any attempt to put arrowheads on the reciprocal diagram to indicate the kinds of stresses in the members results in nothing but confusion. If the kind of stress in a member cannot be discovered by the eye, the polygon of forces for the joint in question must be drawn, as was done in connection with Fig. 98.

70. Rule for Determining the Kind of Stress in a Member. — If, after determining the directions of the stresses in all the

members by means of the reciprocals of the joints, and after marking the ends of the members with arrowheads accordingly, it is found that the arrowheads of a member point toward its joints, the member is then in compression and is a strut; if they point away from the joint the member is in tension and is a tie. Thus it is found that the rafters AD and BD, Fig. 96, are in compression, and the member CD is in tension.

71. Method of Sections in Determining Stresses. — This method depends upon the principle demonstrated in Art. 58, that at any imaginary section of a frame there is equilibrium between the external forces on one side of the section and the stress forces in members on the same side that are cut by the section, and that, therefore, the algebraic sum of the moments about any point in the plane of the frame must be zero. There must, of course, be but one unknown force in the equation of moments, and this will be the case:

(*a*) When only two members are cut and one of them passes through the center of moments.

(*b*) When three members are cut and the center of moments is taken at the intersection of two of them.

(*c*) When the stresses are known in all the members cut except one.

For example, the roof truss of Fig. 98 is reproduced in Fig. 100. Knowing the magnitude and direction of the left reaction R_1, and considering the equilibrium of the joint at the left support, it is seen from inspection that AF is in compression and FE in tension, and they are so marked with arrowheads in the frame.

There being equilibrium at the joint, we have the static equations, $S_1 \cos 30° = S_2$ and $S_1 \sin 30° = R_1$, in which S_1 and S_2 are the stresses in AF and FE respectively. From these equations we get

S_1 = stress in AF = 8 tons, and S_2 = stress in FE = 6.9 tons.

Knowing the stress in AF and the external force AB, an inspection of joint $ABGF$ shows BG and GF to be struts, and they are so marked in the frame. The section xy cuts the members BG, GF, and FE, two of these members intersecting at E. Considering the part of the truss to the left of xy, and denoting the stress in BG by S_3, we have, by moments about the joint at the middle of the lower chord,

$$S_3 \times 12.5 \sin 30^\circ + 2 \times 6.25 = 4 \times 12.5, \text{ whence } S_3 = 6 \text{ tons.}$$

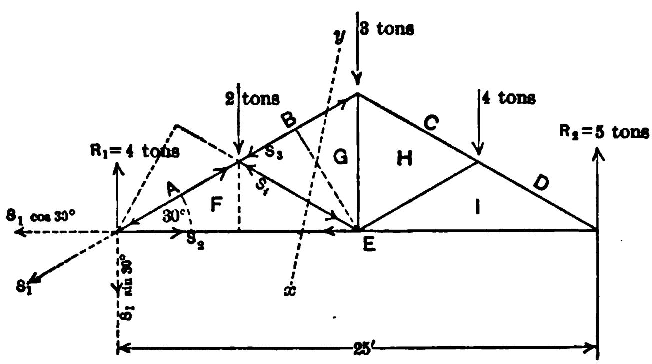


Fig. 100.

Calling S_4 the stress in GF, and taking moments about the left support, we have

$$S_4 \times 12.5 \sin 30^\circ = 2 \times 6.25, \text{ whence } S_4 = 2 \text{ tons.}$$

These results are the same as were obtained from the reciprocal diagram of Fig. 99.

In the examples of the section method just given the nature of the stress, whether tension or compression, in the member whose stress was sought was known, so that the nature of the moments, whether clockwise or contraclockwise, about the center of moments was known. It is not, however, essential that the nature of the stress in a member be known in order to

determine it. A member may be assumed to be either in tension or in compression and the equation of moments written accordingly. Should the solution of the equation give a positive result for the stress, the assumption of its nature is the correct one; should the resulting stress be negative in sign, then the assumption as to the nature of the stress is incorrect, but the numerical value of the stress wili be correct and the same as in the first instance. These conditions arise from the fact that, in the first instance, the moment of the required stress was placed in the proper member of the equation of moments; in the second instance it was improperly placed.

For example, in finding the stress in BG we will assume it to be one of tension. The equation of moments about the middle of the lower chord as a center will be

$$4 \times 12.5 + S_3 \times 12.5 \sin 30^\circ = 2 \times 6.25, \text{ whence } S_3 = -6 \text{ tons},$$

a negative result, which shows that BG is in compression and not in tension, as was assumed, but the numerical result is the same as found above.

Again, assuming FE to be in tension, we shall have, with moments about the joint at the middle of the left rafter,

$$S_2 \times 6.25 \tan 30^\circ = 4 \times 6.25, \text{ whence } S_2 = 6.9 \text{ tons},$$

as was found above, the positive result showing that FE was correctly assumed to be in tension.

72. Stresses in Braced Cantilevers. — The stresses in the members of the braced cantilever of Fig. 101, having a concentrated load of 2 tons at its outer extremity, may be found as follows:

To the scale of $\frac{1}{40}$ inch $= \frac{1}{10}$ ton lay off ab vertical and $\frac{1}{2}$ inch in length to represent the load of 2 tons. From b draw a parallel to BC and from a a parallel to CA. They intersect at c. Then, abc is the triangle of forces for the equilibrium of the joint ABC. The force AB, acting downward, is denoted in direction and

magnitude by *ab* in the triangle, and the stresses in the other members taken in order act in the directions *bc* and *ca*, and their magnitudes are, of course, denoted by the lengths of these lines to the chosen scale.

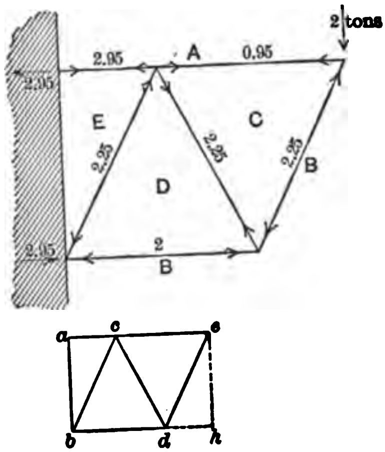


Fig. 101.

Mark these directions by arrowheads at the joint *ABC*. The directions of the stress in the upper end of *BC* and in the outer end of *CA* being now known, it follows that the stresses in the other ends of these members are equal and opposite, and arrowheads must be at once placed to indicate them. We now have sufficient data to draw the force diagram of the joint *CBD*. The stress in the lower end of *CB* has just been marked to act in the direction *cb*. From *b* and *c* draw parallels to *BD* and *DC* respectively, intersecting at *d*; then the triangle *cbd* is the force diagram for the joint *CBD*, and *bd* and *dc* are the directions of the stresses in the members *BD* and *DC*, and they have been marked accordingly in the frame diagram. Arrowheads are at once placed at the other ends of *BD* and *DC* to denote the directions of the equal and opposite stresses at the wall and at

the joint *ACDE* respectively. The stresses in *AC* and *CD* at the joint *ACDE* are now known and their directions are *ac* and *cd* respectively. From *d* draw a parallel to *DE* and from *a* a parallel to *EA*. They intersect at *e*, and *de* and *ea* are the directions and magnitudes of the stresses in the members *DE* and *EA* respectively. It should be noted that, in the reciprocal diagram, the stress line of one member may lie wholly or partly on the stress line of another member, as was here instanced in the stresses of *AC* and *EA*. The stresses in the different members were measured to scale on the reciprocal diagram and marked on the members of the frame.

The horizontal outward pull of 2.95 tons at the upper joint at the wall occasions an equal and opposite reaction. At the lower wall joint there is a horizontal thrust of 2 tons and a diagonal thrust of 2.25 tons. The horizontal component of this diagonal thrust is $hd = ac = 0.95$ ton, making the two reactions equal but opposite in direction.

The braced cantilever of Fig. 102 is 25 feet long, 10 feet deep and uniformly loaded on the top with 100 pounds per foot run.

In apportioning the total load of 2500 pounds, it will be observed that the member *BJ*, being but half the length of each of the members *CH* and *DF*, sustains but one-fifth of the total load, *CH* and *DF* each sustaining two-fifths.

The apportionment of the load will therefore be: 500 pounds at the joint at the outer end, 1000 pounds at the joint *CDFGH*, 750 pounds at the joint *BCHIJ*, and 250 pounds at the joint *ABJ* at the wall. This latter load has no influence on the stresses in the members and is rejected.

Commencing at the joint *DEF*, and with a scale of $\frac{1}{20}$ inch = 100 pounds, we get *def* as the force diagram for the equilibrium of the joint, and at once mark arrowheads on the frame indicating the directions of the stresses *EF*, *FD*, *FE*, and *DF*. Having now the magnitude of the stress *FE* at the joint *FEG*, we obtain

feg as the force diagram of the joint. Marking with arrowheads on the frame the directions of the stresses thus obtained, we find that at the joint *CDFGH* the force *CD* and the stresses in *DF* and *FG* are known, and therefore the force polygon *cdfgh* is readily obtained. The reciprocal diagram may now be completed without difficulty.

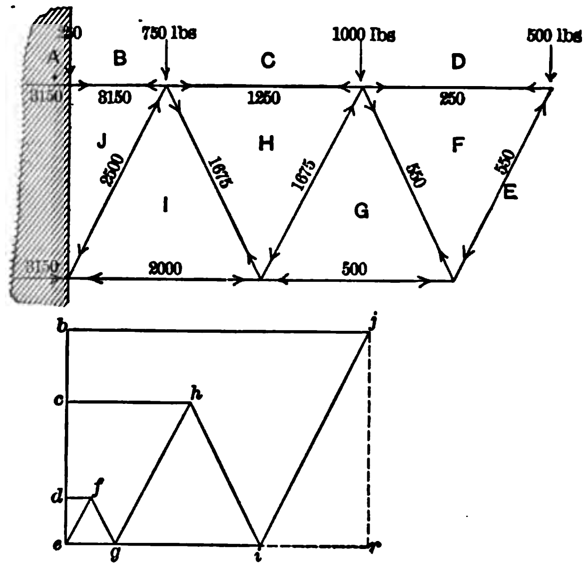


Fig. 102.

73. **The Warren girder** of Fig. 103 has a span of 27 feet, and is loaded with 1⅓ tons per foot, making a total load of 36 tons. The members *AF* and *DL* each sustains but one-sixth of the load, and, in the apportionment, one-twelfth, or 3 tons, falls over the support at each end, and the remainder of the load as shown.

At each end of the top flange there is equilibrium under the action of the external force of 3 tons in line with the upright member and the stresses in the two members at right angles to each other. Evidently the stress in each of the uprights is

3 tons, therefore there cannot be any stress in the members *AF* and *DL* if the equilibrium is to be maintained. To indicate that there is no stress in *AF* the letters *a* and *f* must be placed at the same point in the reciprocal diagram, and because of the absence of stress in *DL* the letters *d* and *l* are at the same point. It should not be forgotten that the two external forces of 3 tons at the ends, coming directly over the supports, are neglected, and

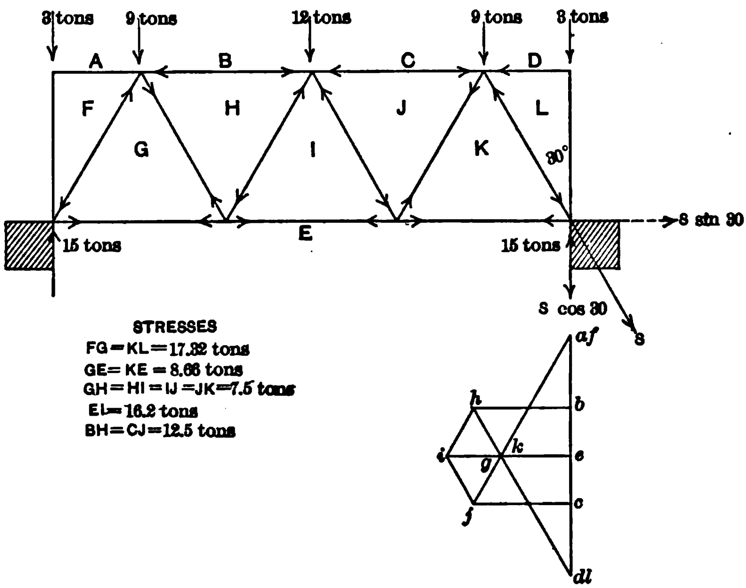


Fig. 103.

that the reactions of 15 tons each are due solely to the loads of 9 tons, 12 tons, and 9 tons, the forces which occasion the stresses in the members. In the construction of the Warren girder the end uprights and the members *AF* and *DL* are omitted.

To the scale of $\frac{1}{8}$ inch = 3 tons set off *ef* to represent in magnitude and direction the left reaction *EF* of 15 tons. From *f* and *e* draw parallels to *FG* and *GE* respectively, intersecting at *g*. Then, *efg* is the triangle of forces for the joint at the left support.

For the joint *GFABH* we get *gfabh* as the polygon of forces, and so on to the completion of the reciprocal diagram.

The triangles of the Warren girder being equilateral, the stresses found from the reciprocal diagram may easily be checked. Thus, denoting the stress in *LK* by *S*, we have

$$S = 15 \sec 30^\circ = 17.32 \text{ tons.}$$

74. The Linville or N girder of Fig. 104 is irregularly loaded on the bottom flange as shown.

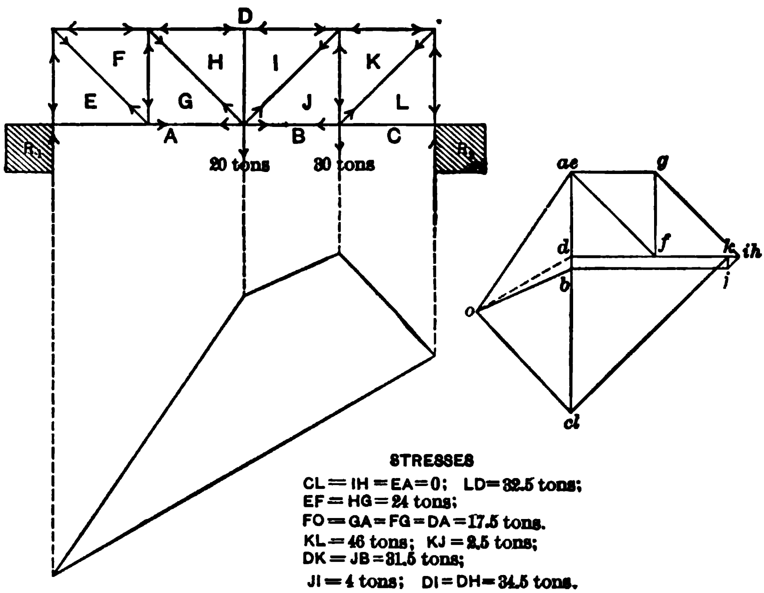


Fig. 104.

For convenience the frame is lettered in this instance in contraclockwise order and the forces at the joints will be considered in like manner.

To a scale of $\frac{1}{40}$ inch = 1 ton, set off *ab* and *bc* to represent in magnitude and direction the loads of 20 tons and 30 tons respectively.

Selecting a pole o we obtain, by means of the funicular polygon, cd and da for the reactions R_2 and R_1 respectively.

For the reason already given there can be no stress in the members CL and EA, and this is indicated by placing l in the reciprocal diagram at the same point as c, and e at the same point as a.

At the joint at the middle of the top flange there is a state of equilibrium under the action of the stresses in the members ID, DH, and HI, HI being at right angles to each of the others. There cannot, therefore, be any stress in HI, and h and i will fall at the same point in the reciprocal diagram.

The members CL and EA add rigidity to the frame, and HI resists the tendency of the top flange to bend.

For the equilibrium of the joint LDK we get the triangle ldk. For the joint $CLKJB$ we get the force polygon $clkjb$, and so on to the completion of the reciprocal diagram.

75. The Fink Truss. — The Fink truss of four bays, Fig. 105, has a span of 64 feet, a depth of 12 feet, and is uniformly loaded with 1.5 tons per foot, making 96 tons in all. The figure is constructed to a scale of $\frac{1}{32}$ inch to the foot, and the load scale is taken as $\frac{1}{80}$ inch to the ton. The apportionment of the load places 12 tons over each support and they are rejected.

In the construction of the reciprocal diagram there are insufficient data to begin at either of the support joints. A consideration of the equilibrium of joint 2 shows the stress in FI to be 24 tons. At joint 11 we have four forces acting along two lines; therefore the conditions of equilibrium require that the stress in FI shall equal that in HG, and that the stress in GF shall equal that in IH. Then the stress in HG is also 24 tons. At joint 10 we have equilibrium under the action of the stresses in GH, HE, and EG; and since HE and EG are equally inclined to GH their stresses must be equal. If, then, the stress in HE can be found, the stress in EG will be known, and there will be

sufficient data to begin at the joint at the left support. Since GH is a strut it is evident that the equilibrium of joint 10 requires that HE and EG be ties, and therefore the direction of the stress in HE at joint 10, and also the direction of the stress in EG at joint 1, are known. By means of the section xy and moments about the left support we have

$$\text{Stress in } HE \times 32 \sin\theta = 24 \times 16,$$

whence, $\quad \text{Stress in } HE = \text{stress in } EG = 20 \text{ tons.}$

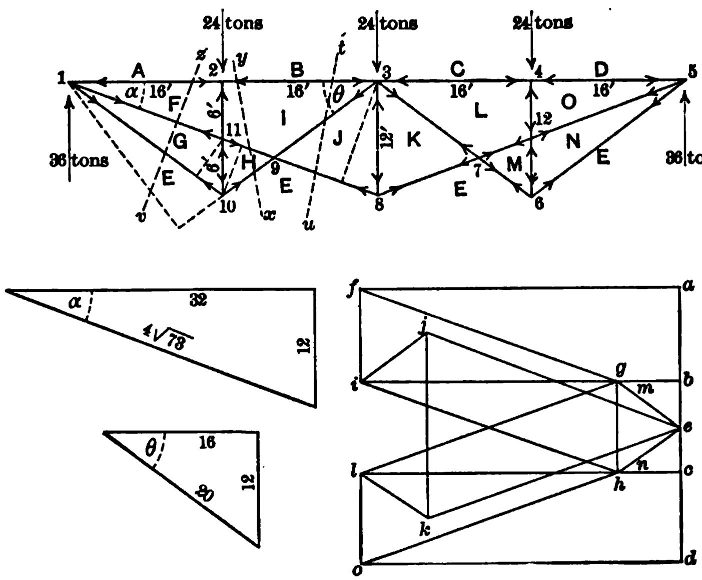


Fig. 105.

There are now sufficient data to begin the construction of the reciprocal diagram by a consideration of the equilibrium of the joint at the left support.

Set off the load line *ad*, making *ab*, *bc*, and *cd* each equal to $\frac{24}{80}$ inch in length to represent the external loads of 24 tons. Then *de* and *ea* are the right and left reactions respectively, to scale.

From e draw eg parallel to EG, and make it $\frac{20}{80}$ inch long to represent the stress of 20 tons in EG. Commencing at e, we get $eafg$ as the stress polygon for the joint at the left support. Knowing af, the stress polygon $abif$ for the joint $ABIF$ is readily drawn, and so on to the completion of the reciprocal diagram.

By means of the section ut and moments about the upper middle joint, we obtain

$$\text{Stress in } JE \times 32 \sin \alpha + 24 \times 16 = 36 \times 32,$$

whence

$$\text{Stress in } JE = \frac{(36 \times 32 - 24 \times 16) \sqrt{73}}{96} = 68.35 \text{ tons},$$

which agrees with the scale length of je of the reciprocal diagram.

The stresses in AF, BI, CL, and DO are shown by the diagram to be 80 tons each. This may be checked by the sections ut and vz.

$$\text{Stress in } BI \times 12 = 36 \times 16 + \text{stress in } JE \times 6 \cos \alpha,$$

whence

$$\text{Stress in } BI = 48 + 68.35 \times \frac{4}{\sqrt{73}} = 80 \text{ tons}.$$

From the section vz and moments about the joint $FIHG$ we have

$$\text{Stress in } EG \times 2\sqrt{73} \sin (\theta - \alpha) + \text{stress in } AF \times 6 = 36 \times 16.$$

whence, $\quad \text{Stress in } AF = \dfrac{576 - 96}{6} = 80 \text{ tons}.$

It will be noted that the Fink truss is composed of a primary truss 1, 8, 5, and two secondary trusses 1, 10, 3, and 3, 6, 5. Whether or not there are joints at 11, 9, 7, and 12 will not affect the stresses in GF, FI, IH, HG, IJ, JE, KL, LM, ME, EK, LO, ON, NM.

76. Roof Truss Fixed at the Ends and with Wind Pressure. — The roof truss of Fig. 106 is fixed at the ends and has the following data:

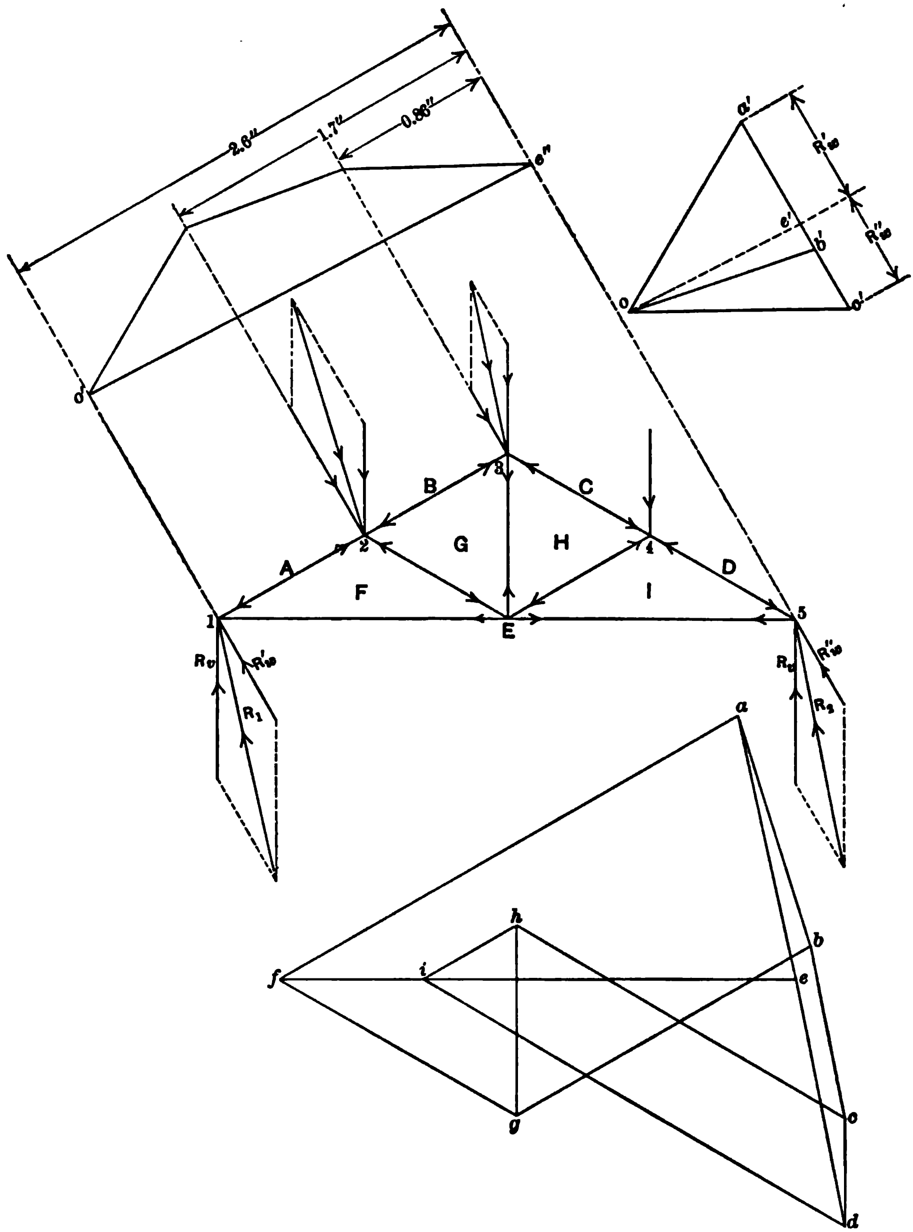


Fig. 106.

Linear scale, 1 inch = 10 feet; load scale, 0.75 inch = 1 ton. Pitch of roof, 30°. Span of roof, 30 feet. Distance between trusses, 10 feet. Dead load per square foot of horizontal surface, 23 pounds. Horizontal wind pressure per square foot, 40 pounds.

The dead vertical load $= \frac{30 \times 10 \times 23}{2240} = 3$ tons, approximately, and its uniform distribution places 0.75 ton at each of the joints 2, 3, and 4, and 0.375 ton at each of the joints 1 and 5.

In this case of a truss *fixed* at the ends, the effect of the wind and of the vertical loads at the end joints is borne entirely by the supports and does not affect the stresses in the members; the loads at the supports are therefore omitted from consideration, and the reactions to be used are those due only to the loads producing the stresses.

The effective support reactions due to the vertical loads are each $\frac{3 - 0.75}{2} = 1.125$ tons $= 1.125 \times 0.75 = \frac{27}{32}$ inch when reduced to scale. These reactions are vertical and are denoted in the frame diagram by R_v.

The vertical loads at the joints 2, 3, and 4 are, when reduced to scale, each equal to $0.75 \times 0.75 = \frac{9}{16}$ inch. These vertical loads are laid off to scale at the joints.

Suppose the wind to act on the left side of the roof. The pitch of the roof being 30°, and the horizontal intensity of the wind pressure being 40 pounds per square foot, the normal pressure is 26 pounds per square foot (see table of Art. 68).

The total wind pressure to be borne by one truss is

$$\frac{15 \sec 30° \times 10 \times 26}{2240} = 2 \text{ tons, very nearly.}$$

The distribution of this wind load places 1 ton at joint 2 and $\frac{1}{2}$ ton at joint 3, the half ton at the left support being rejected. These loads to scale are $\frac{3}{4}$ inch and $\frac{3}{8}$ inch respectively, and are

combined with the vertical loads so as to obtain the magnitudes and directions of the resultant pressures at the joints.

The truss being fixed at the ends the support reactions due to the wind pressure must be parallel to and opposite in direction to the normal wind pressure in order that there shall be a balance between the external forces. The magnitude of these effective wind reactions can be found by means of a funicular polygon. Thus:

Lay off the load line $a'c'$ parallel to the normal wind pressure, making $a'b'$ equal to the wind pressure at joint 2, and $b'c'$ equal to the wind pressure at joint 3. Select at random a pole o, and draw the vectors oa', ob', and oc'. From some point o' in the line of action of the left support reaction construct the funicular of the force diagram $oa'c'$. Draw oe' parallel to the closing line $e''o'$ of the funicular. Then, $e'a'$ is the magnitude of the left support reaction R_w' due to the wind, and $c'e'$ is the magnitude of the right support reaction R_w'' due to the wind. The resultant R_1 of R_v and R_w', and the resultant R_2 of R_v and R_w'' are the total support reactions in magnitude and direction due to the dead load and the wind pressure.

Having the magnitudes and directions of the resultant forces at all the joints, the reciprocal diagram can now be drawn. Thus:

Draw ab, bc, and cd parallel and equal to the resultant forces at joints 2, 3, and 4. From d draw de parallel and equal to R_2. The system being in equilibrium the force polygon must close; therefore the space ea must be exactly filled by a line parallel and equal to R_1; otherwise the construction would be inaccurate.

Commencing at the joint at the left support its reciprocal $eafe$ is readily constructed. Proceeding then to joints 2, 3, 4, and 5 the stresses in all the members are obtained, the necessity of the point i falling on fe affording a check as to accuracy.

The support reactions due to the wind might have been found easily by moments. The frame diagram being a scale drawing, the distances between the lines of action of the normal wind forces and of the reactions due to them are found by measurement to be as shown in Fig. 106. Then, taking moments about the right support, we have

$$26\,R_w' = 1 \times 1.7 + \tfrac{1}{2} \times 0.86, \quad \text{whence} \quad R_w' = 0.82 \text{ ton};$$

hence, $R_w' = 0.82 \times 0.75 = 0.615$ inch $= e'a'$.

Similarly, by moments about the left support, we have

$$26\,R_w'' = 1 \times 0.9 + \tfrac{1}{2} \times 1.74, \quad \text{whence} \quad R_w'' = 0.68 \text{ ton};$$

hence, $R_w'' = 0.68 \times 0.75 = 0.51$ inch $= c'e'$.

The stresses in the members, by scale measurements from the reciprocal diagram, are found to be as here tabulated:

In $EF = 3.57$ tons		Ties
" $GH = 1.30$ "		
" $IE = 2.57$ "		
" $AF = 3.67$ "		Struts
" $BG = 2.33$ "		
" $CH = 2.63$ "		
" $DI = 3.37$ "		
" $IH = 0.73$ "		
" $FG = 1.90$ "		

The solution of this problem with the consideration of all the forces, as shown in Fig. 107, rejecting none at the supports, is instructive.

Lay off the load line *abcdefghi* to scale. Each of the reactions R_v is equal to 1.5 tons $= \frac{3}{2} \times \frac{3}{4} = \frac{9}{8}$ inch to scale. The effective wind reactions are found by the funicular as before, but the left reaction thus found must be augmented by the wind load AB

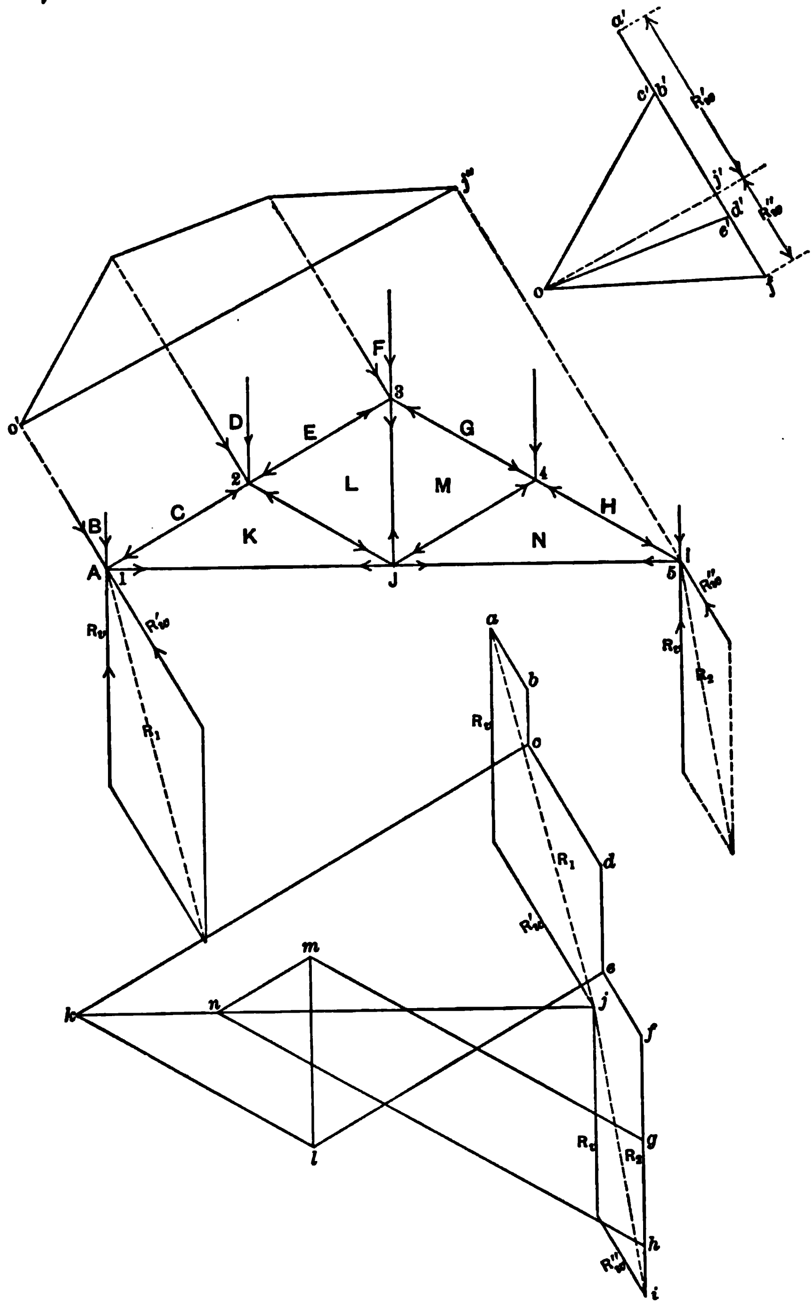

a'
c'
b'
j'
d'
e'
o
f
j''
F
D
E
G
3
2
4
L
M
o'
B
C
K
N
H
A
1
J
5
R_v
R'_w
R''_w
R_1
R_2
a
b
c
d
e
f
g
h
i
j
k
l
m
n

Fig. 107.

in order to obtain the full reaction R_w' at the left support due to the wind. Commencing at *i* the reactions R_w'', R_v, R_w', R_v are each set off in magnitude and parallel direction, closing the force polygon at *a*.

The reciprocal diagram can now be drawn, the stresses obtained being exactly the same as by the preceding method. The dotted lines *aj* and *ji* are the resultant support reactions R_1 and R_2 in magnitude and parallel direction.

77. Roof Truss Fixed at One End and Free at the Other with Wind Pressure and Dead Vertical Loads. — The roof truss of Fig. 108 has a span of 50 feet, and is fixed at one end and free to move on expansion rollers at the other. Pitch of roof, 28°; distance between trusses, 16 feet; dead load per square foot of horizontal surface, 17.9 pounds; wind pressure normal to the truss, 19.77 pounds per square foot. Linear scale, $\frac{1}{16}$ inch = 1 foot; load scale, $\frac{1}{2}$ inch = 1 ton. The frame diagram shows the angular arrangement of the truss members, and it will be noticed that the joints divide the rafters into three parts whose lengths are to each other as 3 : 3 : 2.

The dead vertical load on each truss $= \frac{50 \times 16 \times 17.9}{2240} = 6.4$ tons, 3.2 tons on each rafter, and its uniform distribution places 0.6 ton at joints 1 and 7, 1.2 tons at joints 2 and 6, 1 ton at joints 3 and 5, and 0.8 ton at joint 4.

The support reactions due to the vertical loads are each $\frac{6.4}{2} = 3.2$ tons $= \frac{32}{10} \times \frac{1}{2} = \frac{32}{20}$ inches when reduced to scale. These reactions are vertical, and each is denoted by R_v in the frame diagram. The scale measurements of the vertical loads at the joints are: $0.6 \times 0.5 = 0.3$ inch at joints 1 and 7; $1.2 \times 0.5 = 0.6$ inch at joints 2 and 6; 0.5 inch at joints 3 and 5; and 0.4 inch at joint 4. These loads are laid off to scale at the joints in the frame diagram.

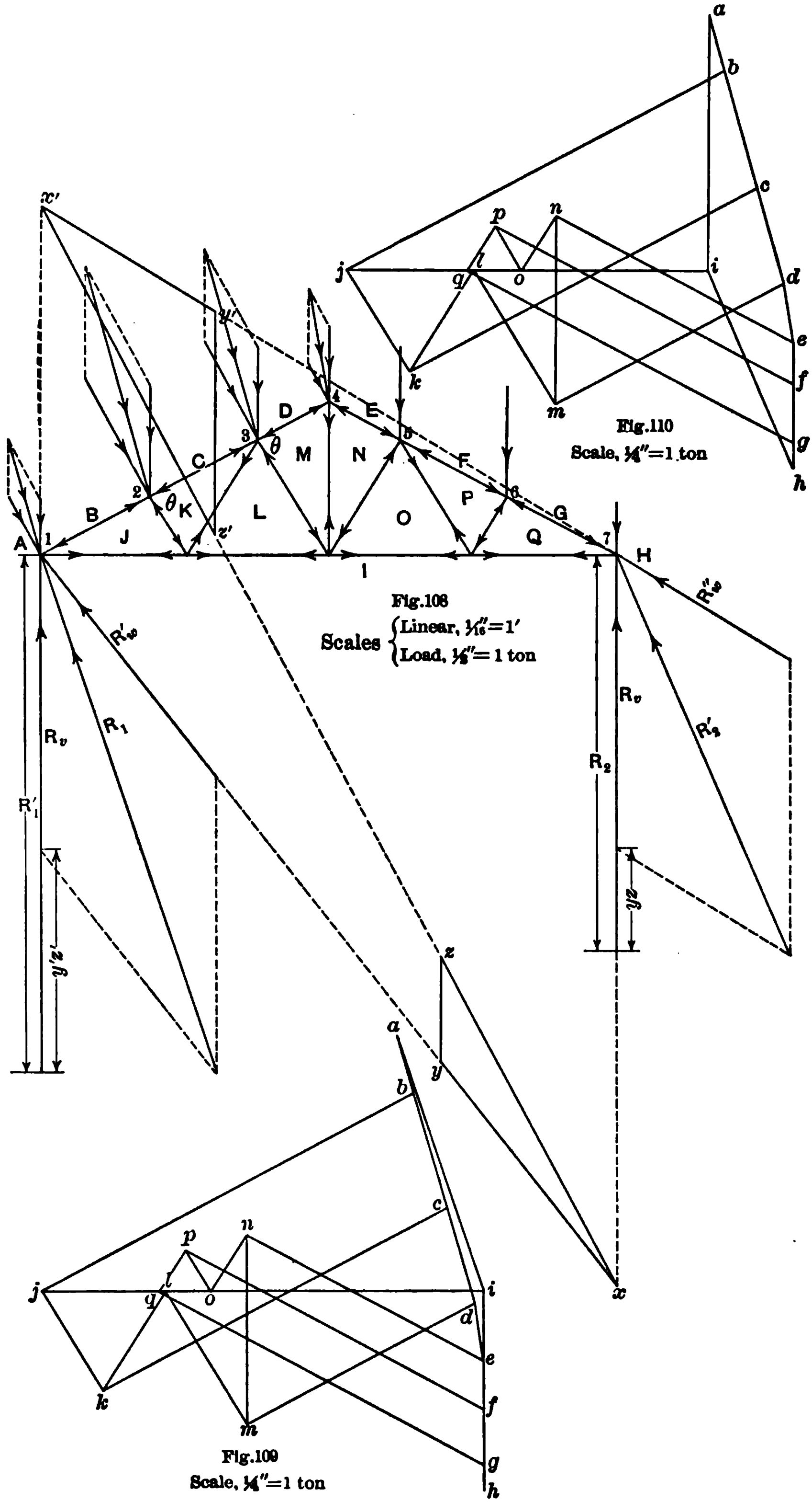


Fig.110
Scale, ⅛″=1 ton

Fig.108
Scales {Linear, 1/16″=1′
Load, ⅛″=1 ton

Fig.109
Scale, ⅛″=1 ton

4)

The total wind pressure to be borne by one truss is

$$\frac{25 \sec 28^\circ \times 16 \times 19.77}{2240} = 4 \text{ tons.}$$

This load may act on either side of the truss. Suppose first that the left end of the truss be fixed and the wind to act on that side. The distribution of the wind load places 0.75 ton at joint 1, 1.5 tons at joint 2, 1.25 tons at joint 3, and 0.5 ton at joint 4. Reduced to scale these loads are: At joint 1, $0.75 \times 0.5 = 0.375 = \frac{3}{8}$ inch; at joint 2, $1.5 \times 0.5 = \frac{3}{4}$ inch; at joint 3, $1.25 \times 0.5 = 0.625 = \frac{5}{8}$ inch; and at joint 4, $0.5 \times 0.5 = 0.25 = \frac{1}{4}$ inch. Set off these loads to scale at the joints in the frame diagram.

The right end being free, the reaction there due to the wind pressure will be vertical.

Assume the truss to be acted on only by the wind and that its resultant pressure of 4 tons acts at the middle point of the left rafter. There are now but three concurrent forces, viz., the two support reactions and the resultant wind pressure. The points of application of these forces and the directions of two of them being known, the direction of the third can be found. Thus:

Let fall a perpendicular to the left rafter from its middle point, and at the right support let fall a perpendicular to the bottom chord of the truss. These perpendiculars are the directions of the resultant normal wind pressure and of the support reaction at the free end respectively, and they intersect at x. The line joining x with joint 1 will give the direction of the support reaction due to the wind at the left (fixed) end. Lay off to scale the distance xz equal to the magnitude, 4 tons, of the resultant normal wind pressure, and resolve it into its components xy and yz parallel to the support reactions. The reaction R_w' at the left support due to the wind is represented in

magnitude by xy. The composition of R_w' and R_v gives the resultant reaction R_1 at the fixed end due to all the external forces. The addition of yz to R_v gives R_2 as the support reaction at the right (free) end due to all the external forces. The resultants of the dead and wind loads at joints 1, 2, 3, and 4 are found by the parallelogram of forces as shown.

The reciprocal diagram for the truss with its left end fixed and the wind blowing on the fixed side can now be drawn.

To a scale of 0.25 inch to the ton — one-half that used for the loads in the frame diagram — construct the force polygon *abcdefghia* of Fig. 109. This polygon must, of course, close. The diagram is readily completed by drawing in order the reciprocals of the joints *IABJ*, *JBCK*, *IJKL*, *LKCDM*, *MDEN*, *ILMNO*, *ONEFP*, and *PFGQ*, the point q falling at the finish on the line ij. The order of procedure from joint to joint determines itself from the fact that the reciprocal of a joint cannot be drawn if more than two of the forces are unknown.

It will be necessary to construct the reciprocal diagram for the wind blowing on the free side and, since the vertical loads are the same on each side, the construction will be facilitated by assuming, in Fig. 108, the right end of the truss to be fixed and the left end free to move on rollers.

The reaction at the left end will now be vertical, and its line of action intersects the line of action of the resultant normal wind pressure on the free side at x', and therefore the line joining x' with joint 7 gives the direction of the reaction at the right support due to the wind. Lay off $x'z'$ equal to 4 tons to scale to represent the magnitude of the resultant wind pressure. Its component $x'y'$ gives the magnitude of R_w'', the reaction at the fixed end due to the wind. The composition of R_w'' and R_v gives R_2' as the support reaction at the right (fixed) end due to all the loads. The component $y'z'$ is the amount to be added to

R_v at the left support to give the total support reaction R_1' at the free end.

The force polygon and the reciprocal diagram can now be drawn as shown in Fig. 110, the point q falling on the line ij.

The members must be designed to resist the maximum stress to which they may be subjected, and since the wind may blow on either side of the truss they are made the same for each side. The tabulated stresses were found by scale measurements from Figs. 109 and 110 to be as follows:

TABLE OF STRESSES.

Members.	Wind on fixed side.		Wind on free side.	
	Ties.	Struts.	Ties.	Struts.
BJ	....	9.20	....	9.25
JI	9.60	....	7.80	....
CK	....	8.50	....	8.50
KJ	....	2.55	....	2.60
KL	2.55	....	2.50	....
LI	6.90	....	5.05	....
DM	....	5.60	....	5.60
ML	....	3.40	....	3.40
EN	....	5.80	....	5.80
NM	4.15	....	4.05	....
NO	....	1.45	....	1.35
OI	5.90	....	4.00	....
FP	....	7.30	....	7.35
PO	1.00	....	1.10	....
GQ	....	7.95	....	8.00
QP	....	1.00	....	1.10
QI	7.00	....	5.20	....

78. A Framed Crane. — To draw the reciprocal diagram of the framed crane of Fig. 111, having a load of 3 tons at its peak, we proceed as follows:

Letter the frame diagram, and then to a scale of 0.25 inch = 1 ton lay off the load line ab three-fourths of an inch in length to represent the load of 3 tons. From b and a draw parallels to BJ and JA respectively. They intersect at j, giving abj as the force diagram of the joint ABJ, and bj and ja represent in

magnitude and direction the stresses in the members BJ and JA respectively. Proceed in like manner to the completion of the

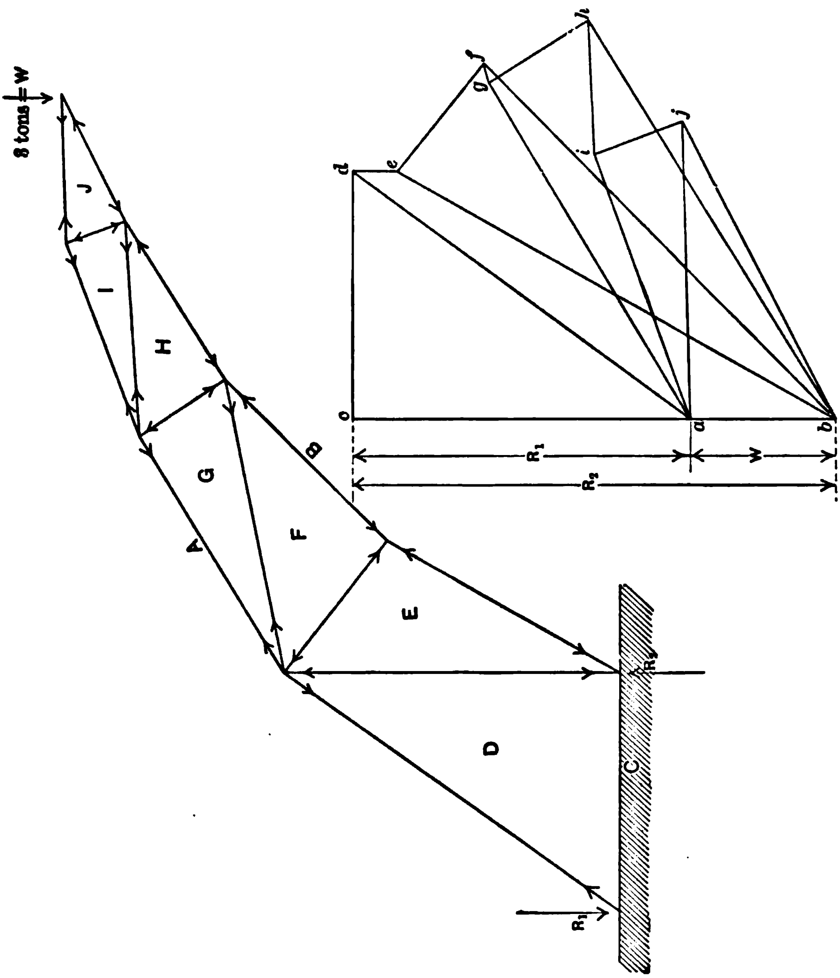


Fig. 111.

reciprocal diagram from which the stress diagram for any joint may at once be read. Thus, the stress diagram for the joint $GHBF$ is $ghbfg$. The reaction R_2 is, of course, equal to $R_1 + W$.

79. Redundant and Deficient Frames. — The reciprocal diagrams for all the frames that have been considered have closed, and the frames, therefore, were complete. It will be noted that in the cantilever frames the number of members equals twice the number of joints minus four, and that in frames supported at the ends the number of members equals twice the number of joints minus three.

If a frame has more than the requisite number of members to retain its original shape, it is called a *redundant* frame and is statically indeterminate.

A frame having an insufficient number of members to preserve its original shape is *deficient*, but may be used for one special distribution of the load. Any deviation from this distribution will cause the frame to change its shape. A deficient frame may generally be made complete by the addition of another member.

PROBLEMS

1. Draw the reciprocal diagram of the Warren girder of Fig. *c*, the triangles being equilateral. Span, 42 feet, and a uniformly distributed load of 1.5 tons per foot is sustained by the girder. Tabulate the stresses and indicate their kinds in the frame diagram. Scales: Load, 0.1 inch = 1 ton; linear, 0.1 inch = 1 foot.

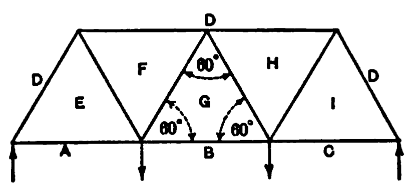


Fig. *c*.

2. Draw the reciprocal diagram of the roof truss of Fig. *d*, which is loaded as shown. Tabulate the stresses and indicate their kinds in the frame diagram. Scales: Load, 0.25 inch = 1 ton; linear, 0.1 inch = 1 foot.

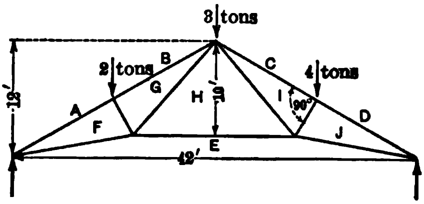


Fig. *d*.

3. The Fink truss of Fig. *e* has a span of 56 feet, a depth of 12 feet, and is uniformly loaded with 1.5 tons per foot. Draw the reciprocal diagram to the scale of $\frac{1}{20}$ inch = 1 ton. Tabulate the stresses and indicate their kinds in the frame diagram. Scale: Linear, $\frac{1}{16}$ inch = 1 foot.

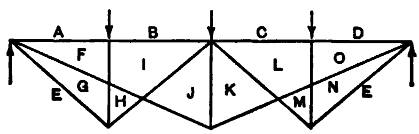


Fig. *e*.

4. The braced cantilever of Fig. *f* is 20 feet long, 9 feet deep, and uniformly loaded on the top with 120 pounds per foot. Draw the reciprocal diagram to the scale of 0.1 inch = 100 pounds. Tabulate the stresses and indicate their kinds in the frame diagram. Show that the reactions at the wall are equal but opposite in direction. Scale: Linear, $\frac{3}{16}$ inch = 1 foot.

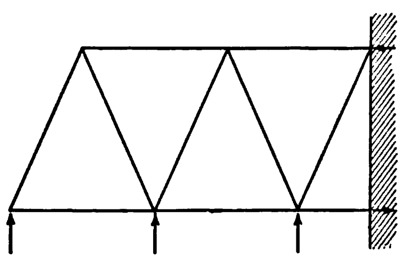


Fig. *f*.

5. Draw the reciprocal diagram of the braced cantilever of Fig. *g*, and tabulate the stresses. Find, in magnitude and direction, the resultant stress on the pin of the upper joint at the wall. Scales: Load, 0.1 inch = 80 pounds; linear, 0.25 inch = 1 foot.

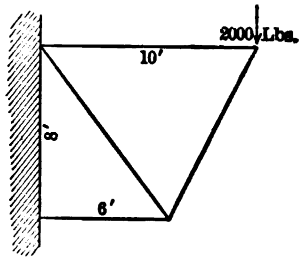


Fig. *g*.

6. A roof truss, Fig. *h*, of 50 feet span, fixed at the ends and of 30° pitch, carries vertical loads of 1.5 tons at *AB*, *BC*, *CD*, *DE* and *EF*, and sustains a horizontal wind pressure of 33.5 pounds per square foot. Distance between principals, 12 feet. Draw the reciprocal diagram and tabulate the stresses in the members. Scales: Linear, $\frac{1}{16}$ inch = 1 foot; load, $\frac{1}{8}$ inch = 1 ton.

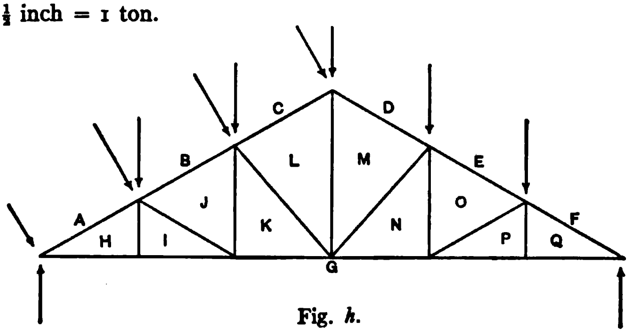


Fig. *h*.

7. The roof truss of Fig. *i* has a span of 75 feet and a pitch of 30°. The rafters are divided into three equal parts and the horizontal tie into five equal parts. The distance between principals is 16 feet. The dead vertical load is 6.5 tons, distributed as follows: At joints 1 and 7, 0.6 ton

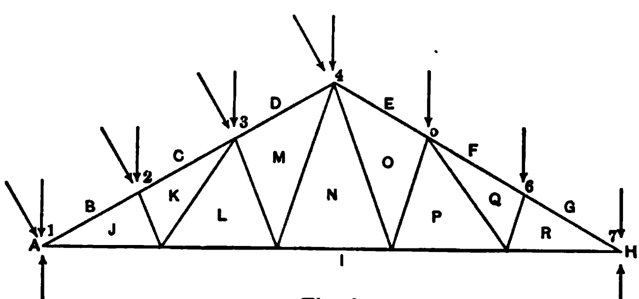


Fig. *i*.

each; at joint 4, 1 ton; at joints 2 and 3, 0.95 ton each; at joints 5 and 6, 1.2 tons each. The wind pressure is assumed to have a horizontal intensity of 20 pounds per square foot. Assuming the truss to be fixed at one end and free to move on expansion rollers at the other, determine the maximum stress in each member. Scales: Linear, $\frac{1}{20}$ inch = 1 foot; load, $\frac{1}{2}$ inch = 1 ton.

8. The roof truss of Fig. *j* has a span of 60 feet, and supports dead loads of 2 tons at each of the joints 2, 3, 4, 5, 6, 7, 8. Draw its reciprocal diagram. Scales: Linear, $\frac{1}{16}$ inch = 1 foot; load, $\frac{1}{2}$ inch = 1 ton.

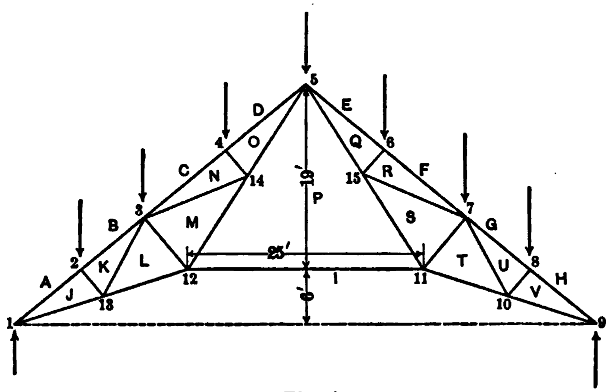


Fig. *j*.

9. A bridge truss, a portion of which is shown in Fig. *k*, has a span of 120 feet divided into six equal bays. It is designed to carry a dead load of 16 tons at each of the lower panel points. Calculate, by the method of sections, the stresses in the members *GM*, *MN*, and *NC*, and indicate by arrowheads the nature of the stresses.

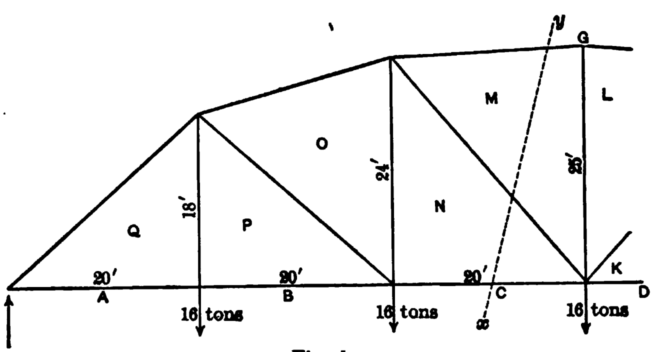


Fig. *k*.

10. Complete the frame diagram of problem 9 and construct the reciprocal diagram, tabulating the magnitudes and kinds of stresses in the members. Scales: Linear, $\frac{1}{10}$ inch = 1 foot; load, $\frac{1}{10}$ inch = 1 ton.

11. The crane of Fig. *l* supports a load of 2 tons at its peak. Determine from the reciprocal diagram the reactions and the stresses in the members. Scales: Linear, $\frac{1}{8}$ inch = 1 foot; load, $\frac{1}{4}$ inch = 1 ton.

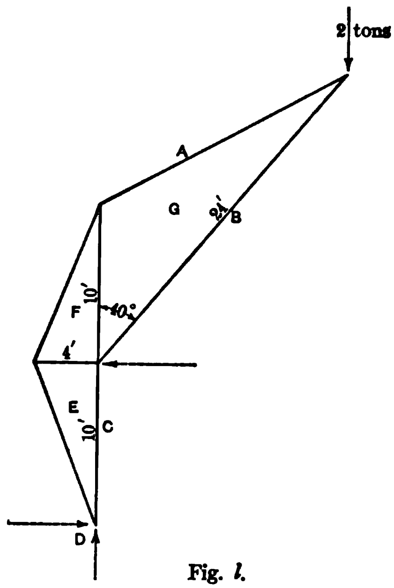


Fig. *l*.

12. Calculate by the method of sections the magnitude and kind of stress in each of the members of the crane of Fig. *m* that is cut by the line *xy*.

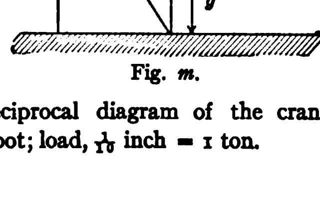


Fig. *m*.

13. Draw the reciprocal diagram of the crane of Fig. *m*. Scales: Linear, $\frac{1}{10}$ inch = 1 foot; load, $\frac{1}{10}$ inch = 1 ton.

CHAPTER IX

ENGINEERING MATERIALS

80. Introductory. — In all engineering construction a knowledge of the properties of the materials used is essential. It is not sufficient to know only the characteristic properties of different materials, but their fitness for use in any engineering design must be determined by actual test in order to insure safety to the structure, and for this purpose the modern mechanical laboratory is equipped with special machines for testing.

The materials used in machine construction must be *strong* and practically *rigid*, the element of strength enabling the size and weight of machines to be reduced to a minimum, and the rigidity enabling the material to resist the tendency to change its size or shape while under the influence of straining actions. As no materials are absolutely rigid, it follows that the materials of machines undergo some deformation under the action of the forces to which they are subjected, but by judicious selection of materials and a proper proportioning of parts, such deformations may be rendered so small as to be negligible. Generally speaking, the deformations in machines are temporary in character, the elasticity of the chosen materials causing them to vanish upon the removal of the straining force.

The principal engineering materials are *cast iron*, *wrought iron*, *steel*, *copper*, *lead*, *tin*, *zinc*, *alloys*, *timber*, *concrete*, and *reinforced concrete*.

81. CAST IRON. — Cast iron is obtained in the form of pig iron by direct treatment of the ore in a blast furnace, its quality

depending on the relative amounts of other substances in its combination.

Influence of Carbon. — In the molten condition the carbon is dissolved by the iron and held in solution just as ordinary salt is dissolved by water. The mixture or combination of the two elements is thus entirely uniform. The proportion of carbon which pure melted iron can thus dissolve and hold in solution is about 3.5 per cent. If chromium or manganese is present also, the capacity for carbon is much increased, while with silicon, on the other hand, the capacity for carbon is decreased. In the various grades of cast iron the proportion of carbon is usually found to be between 2 per cent and 4.5 per cent.

When such a molten mixture cools and becomes solid there is a tendency for a part of the carbon to be separated and no longer remain in intimate combination with the iron. The carbon thus separated, or precipitated, from the iron takes that form known as graphite and collects in very small flakes or scales. The carbon which remains in intimate combination with the iron is said to be combined, while that which is separated is usually called *graphitic*.

The qualities of cast iron depend chiefly on the proportion of total carbon and on the relative proportion of combined and graphitic carbon.

With a high proportion of graphitic carbon the iron is soft and tough, with a low tensile strength, and breaks with a dark and coarse-grained fracture. In fact, the substance in this condition may be considered as nearly pure iron with fine flakes of graphite entangled and distributed through it, thus giving to the iron a spongy structure. The iron thus forms a kind of continuous mesh about the graphite, which decreases the strength by reason of the decrease of cross-sectional area actually occupied by the iron itself. Such irons are termed *gray*.

As the relative proportion of graphitic carbon decreases and

that of combined carbon increases, the iron takes on new properties, becoming harder and more brittle. Its tensile strength also increases to a certain extent, and the fracture becomes fine-grained, or smooth, and whiter in color. When these characteristics are pronounced, the iron is said to be *white*. When about half the carbon is combined and half separates as graphite the effect is to produce a distribution of dark spots scattered over a whitish field. Such iron is said to be *mottled*.

In a general way, with a large proportion of total carbon there is likely to be formed a considerable amount of graphitic carbon, and hence such iron is usually gray and soft. Also, with a large proportion of carbon the iron melts more readily and its fluidity is more pronounced. As the proportion of total carbon decreases the cast iron gradually approaches the condition of steel, whose properties will be discussed in later paragraphs.

Of the special ingredients in cast iron the combined carbon is the one of greatest importance. It is that chiefly which, by uniting with the iron, gives it new qualities, and the principal influence of other substances lies in the effect which they may have on the proportion of this ingredient. As between graphitic and combined carbon the former does not affect the quality of the iron itself, but acts physically by affecting the structure of the casting, while the latter by entering into combination with the iron acts chemically, and produces a new substance with different qualities.

The proportions of combined and graphitic carbon are influenced by the rate of cooling and by the presence or absence of various other ingredients. Slow cooling allows time for the separation of the carbon and thus tends to form graphitic carbon and soft gray irons. Quick cooling, or chilling in the extreme case, prevents the formation of graphitic carbon and thus tends to form hard, white iron.

In addition to carbon, small particles of silicon, sulphur, phosphorus, manganese, and chromium may be found in cast iron.

Influence of Silicon. — The fundamental influences of silicon are two. (*a*) It tends to expel the carbon from the combined state and thus to decrease the relative proportion of combined carbon and increase that of graphitic carbon. (*b*) Of itself, silicon tends to harden cast iron and to make it brittle. These two influences are opposite in character, since an increase in graphitic carbon softens the iron. In usual cases the net result is a softening of the iron, an increase in fluidity, and a general change toward those qualities possessed by iron with a high proportion of graphitic carbon. This applies with a proportion of silicon from 2 per cent to 4 per cent. With more than this the influence on the carbon is but slight and the result on the iron is to decrease the strength and toughness, giving a hard but brittle and weak grade of iron.

A chilled cast iron is an iron which, if cooled slowly, would be gray and soft, but if cooled suddenly by contact with a metal mold, or by other means, becomes white or hard, especially at and near the surface. Certain grades of cast iron tend to chill when cast in sand molds. This property is usually undesirable. In such cases the tendency may be prevented by the addition of silicon, which, by forcing the carbon into the graphitic state on cooling, prevents the formation of hard, chilled surfaces. In all cases the actual effect of adding silicon will depend much on the character of the iron used as a base, and only a statement of the general tendencies can here be given.

To sum up, a white iron which would give hard, brittle, and porous castings can be made softer, tougher, and more solid by the addition of silicon to the extent of perhaps 2 per cent or 3 per cent. As the silicon is increased, the iron will become softer and grayer and the tensile strength will decrease. At the same time the shrinkage will decrease, at least for a time, though it

may increase again with large excess of silicon. The softening and toughening influence, however, will only continue so long as additional graphite is formed, and when most of the carbon is brought into this state the maximum effect has been produced and any further addition of silicon will decrease both strength and toughness.

Influence of Sulphur. — Authorities are not in entire agreement as to the influence of sulphur on cast iron, some believing that it tends to increase the proportion of combined carbon, while others maintain that it tends to decrease both the combined carbon and the silicon. It is generally agreed, however, that in proportions greater than about 0.15 to 0.20 of 1 per cent it increases the shrinkage and the tendency to chill and decreases the strength. Sulphur does not readily enter cast iron under ordinary conditions and its influence is not especially feared. An increase in the proportion of sulphur in cast iron is most likely to result from an absorption of sulphur in the coke during the operation of melting in the cupola.

Influence of Phosphorus. — The presence of phosphorus increases the fusibility, fluidity, and brittleness of cast iron and is desirable in light and ornamental castings. The maximum amount of phosphorus should not exceed 1.5 per cent.

Influence of Manganese. — This element by itself decreases fluidity, increases shrinkage, and makes the iron harder and more brittle. It combines with iron in all proportions. The combination containing less than 50 per cent of manganese is called *spiegeleisen.* With more than 50 per cent of manganese it is called *ferromanganese.* One of the most important properties of manganese in combination with iron is that it increases the capacity of the iron for carbon. Pure iron will take only about 3.5 per cent of carbon, while with the addition of manganese the proportion may rise to 6 or 7 per cent. Manganese is also believed to decrease the capacity of iron for sulphur and to this

extent may be a desirable ingredient in proportions not exceeding 1 to 1.5 per cent.

Shrinkage. — At the moment of hardening, cast iron expands and takes a good impression of the mold. In the gradual cooling after setting, however, the metal contracts, so that on the whole there is a shrinkage of about 0.125 inch per foot in all directions, though this amount varies somewhat with the quality of the iron and with the form and dimensions of the pattern. In a general way, hardness and shrinkage increase and decrease together.

Strength and Hardness. — The strength of cast iron is chiefly dependent upon its amount of combined carbon. The greatest crushing strength is obtained with sufficient combined carbon to make a rather hard, white iron, while for the maximum transverse or bending strength the combined carbon is somewhat less and the iron only moderately hard. For the greatest tensile strength, the combined carbon is still less and the iron rather soft. Metal still softer than this works with the greatest facility but is deficient in strength.

Uses in Engineering. — Cast iron is used for cylinders, cylinder heads, liners, slide valves, valve chests and connections, and generally for all parts having considerable complexity of form. It is also used for columns, bed plates, bearing pedestals, caps, etc., though cast and forged steel are to some extent displacing cast iron for some of these uses. It is also used for grate bars, furnace door frames, and for minor boiler fittings.

Inspection of Castings. — In the inspection of castings care must be had to note the texture of the surface and to this end the outer scale and burnt sand should be carefully removed by the use of brushes or chipping hammer, or, if necessary, by pickling in dilute muriatic acid. The flaws most liable to occur are blow holes and shrinkage cracks. The parts of the casting most liable to be affected by blow holes are those on the upper

side or near the top, and on this account a sinking head or extra piece is often cast on top, into which the gases and impurities may collect. This is afterwards cut off, leaving the sounder metal below. Shrinkage cracks are of unusual occurrence. The presence of blow holes, if large in size or in great number and near the surface, may often be determined by tapping with a hammer. The sound given out will serve to indicate to an experienced ear the probable character of the metal underneath.

Brazing of Cast Iron. — Cast iron may be brazed to itself, or to most of the structural metals, by the use of a brazing solder of suitable melting point and with proper care in the operation. Cast iron may also be united to itself, or to wrought iron or steel, by the operation of burning. This consists in placing in position the two pieces to be united, and then allowing a stream of melted cast iron to flow over the surfaces to be joined, the adjacent parts being protected by fire clay or other suitable material. The result is to soften or partially melt the surfaces of the pieces, and by arresting the operation at the right moment they may be securely joined together.

Malleable Cast Iron. — If iron castings of not too great thickness, and of such purity as to be low in sulphur, be embedded in powdered red oxide of iron (red hematite) and maintained at a red heat for two or three days, they become, in a measure, decarbonized as a result of the chemical action which ensues. The carbon first disappears from the outer layers, and as the process continues decarbonization toward the innermost layers takes place. If the process be carried to the extreme in the effort to withdraw all the carbon from the interior the outer layer is very liable to become brittle, thus defeating the object to be attained. For this reason there always remains a core of cast iron only partially decarbonized, while the outermost layers are left in the condition of soft or malleable iron. The process has little effect upon the sulphur, manganese, phosphorus, and

other impurities of the castings, but the resulting product is a malleable casting much less fusible than cast iron and possessing six times its ductility. The best product may be twisted and bent to a considerable extent before breaking, and its ability to withstand shocks is much greater than that of cast iron. Pipe fittings, to some extent, are malleable castings, and this material is largely used in appliances of car construction where more strength and toughness are required than cast iron affords.

82. **Wrought Iron.** — Wrought iron is nearly pure iron mixed with more or less slag. Nearly all the wrought iron used in modern times is made from cast iron by the puddling process in a reverberatory furnace. For the details of this process reference may be had to textbooks on metallurgy. We can only note here that, in a furnace somewhat similar to the open hearth, most of the carbon, silicon, and other special ingredients of cast iron are removed by the combined action of the flame and of a molten bath of slag or fluxing material, consisting chiefly of black oxide of iron. As this process approaches completion small bits of nearly pure iron separate from the bath of melted slag and unite. This operation is assisted by the puddling bar, and after the iron has thus become separated from the liquid slag it is taken out, hammered or squeezed, and rolled into bars or plates. Some of the slag is necessarily retained in the iron and, by the process of manufacture, is drawn out into fine threads, giving to the iron a stringy or fibrous appearance when nicked and bent over or pulled apart.

The proportion of carbon in wrought iron is very small, ranging from 0.02 to 0.20 of 1 per cent. In addition, small amounts of sulphur, phosphorus, silicon, and manganese are usually present.

The proportion of sulphur should not exceed 0.01 of 1 per cent. Excess of sulphur makes the iron *red-short*, that is, brittle when red hot.

The proportion of phosphorus may vary from 0.05 to 0.25 of 1 per cent. Excess of phosphorus makes the metal *cold-short*, that is, brittle when cold.

The proportion of silicon may vary from 0.05 to 0.30 of 1 per cent.

The proportion of manganese may vary from 0.005 to 0.05 of 1 per cent. The influence of the silicon and manganese is usually slight and unimportant.

Special Properties. — Wrought iron is malleable and ductile, and may be rolled, flanged, forged, and welded. It cannot be hardened, though by the process of case-hardening a surface layer of steel is formed which may be hardened. Wrought iron may be welded, because for a considerable range of temperature below melting (which takes place only at a very high temperature) the iron becomes soft and plastic, and two pieces pressed together in this condition unite and form, on cooling, a junction nearly as strong as the solid metal. In order that this welding operation may be successful, the iron must be heated sufficiently to bring it to the plastic condition, yet not overheated, and there must be employed a flux (usually borax) which will unite with the iron oxide and other impurities at the joint and form a thin liquid slag which may readily be pressed out in the operation, thus allowing the clean metal surfaces of the iron to effect a union as desired.

83. STEEL. — Steel may be made from wrought iron by increasing its proportion of carbon, or from cast iron by decreasing its proportion of carbon. The earlier processes followed the first method, and high-grade steels are still made in this way by the *crucible process*.

The properties of steel depend partly on the proportions of carbon and other ingredients which it may contain, and partly on the process of manufacture. The proportion of carbon is intermediate between that for wrought iron and for cast iron.

In the so-called mild or structural steel the carbon is usually from 0.1 to 0.3 of 1 per cent. In spring steel the carbon proportion is somewhat greater, and in high carbon grades, such as are used for tools, etc., the carbon is from 0.6 to 1.2 per cent. In addition to the carbon there may be sulphur, phosphorus, silicon, and manganese in varying but very small amounts.

Crucible Steel. — In the crucible process of making steel a pure grade of wrought iron is rolled into flat bars. These are then cut, piled, and packed with intermediate layers of charcoal and subjected to a high temperature for several days. This recarbonizes or adds carbon to the wrought iron and thus makes what is called *cement* or *blister* steel. These bars are then broken into pieces of convenient size, placed in small crucibles, melted, and cast into bars, or into such shapes as are desired.

Structural Steel. — Structural or mild steel is made by the second general process, that of reducing the proportion of carbon in cast iron. In this operation there are two processes, known as the *Bessemer* and the *Siemens-Martin* or *open hearth.*

Bessemer Process. — In this process the carbon and silicon are burned almost entirely out of the cast iron by forcing an air blast through the molten iron in a vessel known as a converter. A small amount of spiegeleisen, or iron rich in carbon and manganese, is then added in such quantity as to make the proportion of carbon and manganese suitable for the charge as a whole. The steel thus formed is then cast into ingots, or into such forms as may be desired. In this process no sulphur or phosphorus is removed, so that it is necessary to use a cast iron nearly free from these ingredients in order that the steel may have the properties desired.

A modification, by means of which the phosphorus is removed, and known as the *basic* Bessemer process, is used to some extent. In this process, calcined or burnt lime is added to the charge just before pouring. This unites with the phosphorus,

removes it from the steel, and brings it into the slag. In the basic process the converter is lined with ganister — a mixture of ground quartz and fire clay — to protect it from attack by the limestone added to the charge and from the resulting slag.

In the Bessemer process first noted, often known as the *acid* process, in distinction from the *basic* process, the lining of the converter is of ordinary fire clay.

The removal of the phosphorus by the basic process makes possible the use of an inferior grade of cast iron. At the same time, engineers are not altogether agreed as to the relative values of the two products, and many prefer steel made by the acid process from an iron nearly free from phosphorus at the start.

The Open-hearth Process. — In this process a charge of material consisting of wrought iron, cast iron, steel scrap, and sometimes certain ores, is melted on the hearth of a reverberatory furnace heated by gas fuel on the Siemens-Martin or regenerative system. The carbon is thus partially burnt out in much the same manner as for wrought iron, and the proportion of carbon is brought down to the desired point, or slightly below that point. A charge of spiegeleisen is then added in order that the manganese may act on any oxide of iron slag which remains in the bath which, if allowed to form a part of the charge, would make the steel red-short. The manganese separates the iron from the oxide and returns it to the bath, while the carbon joins with that already present and thus produces the desired proportions.

Here, as with the similar operations with the Bessemer converter, there is no removal of sulphur or of phosphorus, and only materials nearly free from these ingredients can be used for steel of satisfactory quality. With very low carbon, however, a little phosphorus seems to be desirable to add strength to the metal. This limitation of the available materials has led, as with

the Bessemer process, to the use of calcined limestone in the charge, its purpose being to unite with most of the phosphorus and hold it in the slag. As with the Bessemer process, it is necessary in this case to use a basic lining for the furnace, and it is known as the *basic open-hearth* process. The process which does not use limestone in the charge has come to be known as the *acid open-hearth* process. There is much difference of opinion as to the relative merits of the two open-hearth processes. Either will produce good steel with proper care and neither will without it. It is usually considered sufficient to specify the allowable limits of phosphorus and sulphur and then leave the choice of the acid or basic process to the maker.

Open-hearth and Bessemer Steels Compared. — Open-hearth steel is usually preferred for structural material, for these reasons:

(*a*) It seems to be more reliable and less subject to unexpected or unexplained failure than the Bessemer product.

(*b*) Analysis shows that it is much more homogeneous in composition than Bessemer steel, and experience shows that it is much more uniform in physical quality. This is due to the process of manufacture, which is much more favorable to a thorough mixing of the charge than in the Bessemer process.

(*c*) The open-hearth steel may be tested from time to time during the operation, so that its composition may be determined and adjusted to fulfill special conditions. This is not possible with the Bessemer process.

Influence of Sulphur. — Sulphur makes steel red-short and interferes with its forging and welding properties. Manganese tends to counteract the bad effects of sulphur. Good crucible steel has rarely more than 0.01 of 1 per cent of sulphur. In structural steel the proportion may vary from 0.02 to 0.10 of 1 per cent. When possible it should be reduced to not more than 0.03 or 0.04 of 1 per cent.

Influence of Phosphorus. — Phosphorus increases the tensile strength and raises the elastic limit of low carbon or structural steel, but at the expense of its ductility and toughness, or ability to withstand shocks and irregularly applied loads. It is thus considered as a dangerous ingredient and the amount allowable should be carefully specified. This is usually placed from 0.02 to 0.10 of 1 per cent.

Influence of Silicon. — Silicon tends to increase the tensile strength and to reduce the ductility of steel. It also increases the soundness of castings and ingots and, by reducing the iron oxide, tends to prevent red-shortness. The process of manufacture usually removes nearly all of the silicon, so that it is not an element likely to give trouble to the steel makers. The proportion allowable should not be more than from 0.1 to 0.2 of 1 per cent.

Influence of Manganese. — This element is believed to increase hardness and fluidity, and to raise the elastic limit and increase the tensile strength. It also removes iron oxide and sulphur and tends to counteract the influence of such amounts of sulphur and phosphorus as may remain. It is thus an important factor in preventing red-shortness. The proportion needed to obtain these valuable effects is usually found between 0.2 and 0.5 of 1 per cent.

Semisteel. — A metal bearing this trade name has in recent years attracted favorable attention and has come into considerable use where somewhat greater strength and toughness are required than can be provided by cast iron. It is made by melting mild steel scrap, such as punchings and clippings of boiler plate, with cast-iron pig in the proportions of 25 or 30 per cent of the former to 75 or 70 per cent of the latter. The presence of manganese and other special fluxes in small proportions is found to add essentially to the strength, toughness, and ability to withstand shocks decidedly greater than for cast

iron, and with fairly good machine qualities. Semisteel casts as readily as most grades of cast iron, and its shrinkage and general manipulation are about the same.

Mechanical Properties. — The tensile strength of the lowest carbon steel, say about 0.10 of 1 per cent carbon, is usually from 50,000 to 55,000 pounds per square inch of section. The strength increases quite uniformly with the increase of carbon, provided there are no unusual proportions of sulphur and phosphorus. Experiment shows that under these circumstances the tensile strength will increase up to 75,000 pounds per square inch, or higher, at the rate of from 1200 to 1500 pounds per 0.01 of 1 per cent of carbon added. At the same time, with the increase in strength the ductility decreases, so that a proper choice must be made according to the particular uses for which the steel is intended. In the best grades of tool steel, with carbon ranging from 0.5 to 1.0 per cent or over, the strength ranges from 80,000 to 120,000 pounds, and even higher in exceptional cases.

Special Properties. — Mild or low carbon steel may be welded, forged, flanged, rolled, and cast. It cannot be tempered or hardened with a proportion of carbon lower than about 0.75 of 1 per cent. High carbon steel can be welded only imperfectly and, if very high in carbon, not at all. It can be forged with care and cast in forms as desired. It can be tempered or hardened by heating to a full yellow and quenching in cold water or by other means, and then drawing the temper to the point desired.

Mild steel should not be worked under the hammer or flanging press at a low, or blue, heat, as such working is found in many cases to leave the metal brittle and unreliable. Steel, in order to weld satisfactorily, should have a low proportion of sulphur, and special care is required in the operation, because the range of temperature through which the metal is plastic and fit for welding is less than with wrought iron.

Tempering. — In the operation of tempering, the steel after quenching is very hard and brittle. In order to give the metal the properties desired, the temper is drawn down by reheating it to a certain temperature and then quenching again; or better still, by allowing it to cool gradually, provided the temper does not rise above the limiting value suitable for the purpose desired. If the reheating is done in a bath of oil, the conditions may be kept under good control and the final cooling may be slow. If the reheating is in or over a fire, the control is lacking and the piece must be quenched as soon as the proper temperature is reached. This is usually determined by the color of the oxide or scale that forms on the brightened surface of the metal. The following table shows the temperatures, the corresponding colors, and the uses for which the different tempers are suited:

430° Faint yellow 450° Straw yellow 470° Full yellow	Hardest and keenest cutting tools.
490° Brown yellow or orange	Cutting tools requiring less hardness and more toughness.
510° Purplish 530° Purple	Tools for softer materials, or those required to stand rough usage.
550° Light blue 560° Full blue 600° Dark blue	Spring temper. Used for tools requiring great elasticity, and those for working very soft materials.

Special Steels. — In the common grades of steel the valuable properties are due to the presence of carbon, modified in some degree by other ingredients. There are other substances which, when united with iron in small proportions, give to the combination increased strength, hardness, or other valuable properties. We have thus various special steels in which their properties may be due to the presence of carbon and other ingredients, or due chiefly to special ingredients other than carbon. The most

important of the special steels are known as *nickel steel* and *tungsten steel.*

Nickel Steel. — An alloy known as nickel steel, containing about 3 per cent of nickel and varying amounts of carbon, is found to have increased strength and toughness as compared with ordinary steel. It is extensively used in the manufacture of guns and armor plate, and to some extent it has been employed in government work for propeller shafts and for boiler plates.

Tungsten Steel. — This steel, known also as *Mushet steel,* containing tungsten in proportions varying from 8 to 15 per cent, is very hard and can be forged only by the exercise of great care. Its hardness is not increased by tempering but is naturally acquired as the metal cools; hence it is said to be self-hardening. Some specimens contain also small amounts of manganese and silver. Its chief use is for lathe, planer, and other cutting and shearing tools where excessive hardness is required.

Uses in Engineering. — Structural steel is used extensively in the construction of buildings and bridges and almost entirely in the construction of the hulls of modern ships.

Cast steel, as well as cast iron, is used for pistons and crossheads of engines, columns, bed plates, bearing pedestals and caps, propeller blades, and for many small pieces and fittings.

Forged steel is used for columns, piston rods, connecting rods, crank and line shafting, and for other parts of engines and machinery.

84. COPPER. — Copper in its pure state is red in color, soft, ductile, and malleable, with a melting point at about 2000 degrees, and a tensile strength of from 20,000 to 30,000 pounds per square inch of section. It is not readily welded except electrically, but is easily joined by the operation of brazing. Attempts have been made to temper it, but without practical

success. It is readily forged and cast, and when cold, may be rolled into sheets or drawn into wire. When in sheets or in small pieces it may be spun, flanged, and worked under the hammer.

The tensile strength of copper rapidly falls off as the temperature rises above 400 degrees, so that from 800 to 900 degrees its strength is only about one-half that at ordinary temperatures. This peculiarity of copper should be borne in mind when it is used in places where the temperature is liable to rise to these figures. If copper is raised nearly to its melting point in contact with air, it readily unites with oxygen and loses its strength in large degree, becoming, when cool, crumbly and brittle. Copper in this condition is said to have been burned. The possibility of thus injuring the tenacity of copper is of the highest importance in connection with the use of brazed joints in steam pipes.

Copper unalloyed is used chiefly for pipes and fittings, especially for junctions, elbows, bends, etc. For large sizes of pipes and fittings, the copper is made in sheets, bent and formed to the desired shape, and brazed at the seams; for small sizes the same general process is followed, or the metal is drawn from the solid and bent as desired after the drawing.

Copper is also used as the chief ingredient of the various brasses and bronzes.

85. **Lead.** — Lead is a very soft, dense metal, grayish in color after exposure to the air, but of a bright silvery luster when freshly cut. Commercial lead often contains small amounts of iron, copper, silver, and antimony, and when so combined is harder than the pure metal. It is very malleable and plastic. In engineering, lead is chiefly of value as an ingredient of bearing metals and other special alloys. Lead piping is also used to some extent as suction and delivery pipes for water where the pressure is only moderate, and where the readiness with which

it may be bent and fitted adapts it for use in contracted places.

86. Tin. — Tin is a soft, white, lustrous metal with great malleability. Commercial tin usually contains small portions of many other substances, such as lead, iron, copper, arsenic, antimony, and bismuth. It is largely used as an alloy in the various bronzes and other special metals. Tin resists corrosion well, and in consequence is frequently used as a coating for condenser tubes. It is also used for coating iron plates, the product being the so-called *tin plate* of commerce. It melts at about 450 degrees, which corresponds to a steam pressure of 400 pounds per square inch, approximately. Due to this low melting point, tin is often used in the composition for safety plugs in boilers.

87. Zinc. — Zinc, or "spelter," as it is often commercially called, is a brittle and moderately hard metal with a very crystalline fracture. The impurities most commonly found in zinc are iron, lead, and arsenic. It is used chiefly as an ingredient of the different brass and bronze alloys, and for coating iron and steel plates and rods. The process of applying zinc for such a coating is called *galvanizing*, and the product is "galvanized" iron or steel. Electricity is not used in the process; the articles, after being well cleaned, are simply dipped into a tank of melted zinc and then withdrawn.

88. Alloys. — A mixture of two or more metals is called an *alloy*. The properties of an alloy are often surprisingly different from those of its ingredients. The melting point is sometimes lower than that of any of the ingredients, while the strength, elastic limit, and hardness are often higher than for any one of them.

Mixtures of copper and zinc are called *brass*. Mixtures of copper and tin, or of copper, tin, and zinc, with sometimes other substances in small proportions, form *gun metals*, *compositions*, and *bronzes*. These terms are rather loosely employed.

Brass and compositions are used for piping and pipe fittings; globe, gate, check, and safety valves; condenser tubes and shells. The bronzes are employed for many of the uses of brass where more hardness, strength, and rigidity are required. They are used with success as a material for propeller blades.

89. TIMBER. — Timber is not extensively used in modern engineering construction. The advance made in the production of steel, whereby its homogeneity is assured and its superior strength unquestioned, has caused it to be substituted for wood whenever it is possible and profitable to do so.

Generally speaking, the heaviest and darkest colored timber is the strongest, and, in all cases, the strength of timber is greatest in the direction of its grain.

The locality from which timber comes, the season of its cutting, and the duration of its seasoning, are factors in its strength, and the uncertainty arising therefrom makes it advisable to use a factor of safety of not less than 10 in calculations relating to the dimensions of timber to bear given loads.

90. CONCRETE. — Concrete is a mechanical mixture of cement, sand, and broken stone or gravel, in the proportions, usually, of 1 : 2 : 4 or 1 : 3 : 6. It is largely used for laying foundations for buildings and bridges in wet ground, and for breakwaters and sea walls. After laying, it soon hardens to a strong mass which is little permeable to water.

Concrete is not in the market as a manufactured product but must be made as needed. Whatever the proportions used, there is the utmost necessity for thorough mixing, water being added as may be necessary to secure coherency in the mixture. The use of machines designed for the purpose secures a more perfect mixture than that attained by the hand process.

91. REINFORCED CONCRETE. — Beams and columns of reinforced concrete are products designed to avoid the uncertainty concerning the protection from corrosion and fire

that attends the use of the skeleton steel frame in building construction.

There can be no question as to the appropriate use of steel for constructive purposes in such open structures as bridges and steamships, but in cases where a steel skeleton is vested with the strength of a structure, and is subsequently incased in terra cotta, stone, or brick, precluding visual inspection, there is serious question as to the propriety of its use.

The use of concrete for constructive purposes was common among the ancients, and the fact that in some ruins, as they stand to-day, the concrete parts remain — the stone having long since disappeared — is conclusive evidence of its durability. There is abundant evidence also that iron embedded in concrete is protected from the corrosive influence of moisture and from the ravages of fire.

It has been demonstrated experimentally that concrete is strong in compression but quite weak in tension. In the case of a concrete beam supported at the ends and loaded at the middle, it has been shown that the upper side, which is in compression, is capable of supporting ten times the load which would cause failure at the lower side, which is in tension.

Having in steel a material of very high tensile strength, and in concrete a material possessing high compressive strength as well as the properties of durability and impermeability to moisture, the problem arose of effecting their combination so as to produce a composite material having the desirable qualities of both. The solution of this problem was the production of reinforced concrete.

In the case of beams, steel bars are embedded in the area of the concrete below the neutral axis to reinforce the concrete subjected to tensile stress, thus enabling the full strength of the compression area above the neutral axis to be utilized in sup-

porting heavier loads than would have been possible without the combination.

In the case of columns, the well-known fact that the most economical metal section is that of the hollow cylinder suggested at once the conception of a column having three concentric parts, viz., a central core of concrete, an intermediate zone of steel, and an outer zone of concrete. The steel is thus protected from fire and moisture and is best disposed for the utilization of its maximum strength.

Between the tubular reinforced concrete column just described and the plain non-reinforced concrete column there are a variety of possible combinations. In the form generally used, applicable alike to columns and piles, the reinforcement consists of vertical rods of steel tied together by a system of horizontal wires. These horizontal ties not only keep the vertical rods in position, but materially assist in preventing flexure in them; they also prevent lateral bulging of the concrete.

The reinforcement for both beams and columns is first placed in position, after which they are enveloped by a wooden form, or mold, into which the concrete is dumped, and rammed when necessary. After a period of thirty hours the form may be removed and the product allowed to season for several weeks before being subjected to its load.

92. **Adhesion of Concrete to Steel.** — It has been shown by experiment that the concrete on the compression side of reinforced concrete beams may be subjected without rupture to a stress twenty times that which would cause failure in a tension test in the concrete alone. It has also been shown that reinforced concrete acquires a power to resist crushing which is greater than the sum of the resistances of the two materials taken separately. Such remarkable results could not be obtained were it not for the adhesion between the concrete and steel, such adhesion offering resistance to sliding between the two surfaces

and facilitating the transference of the forces from one surface to the other.

The bond between the steel and concrete occasioned by adhesion alone is liable to be destroyed by internal stresses due to shocks and vibrations, and from unequal expansions resulting from thermal changes, and it is with the idea of strengthening the adhesion bond by mechanical means that the reinforcing bars are usually twisted or have projections on their surfaces.

93. **Proportion of Reinforcement.** — To secure a uniform distribution of the stress in a reinforced concrete section, the reinforcement should consist of steel bars of small section distributed so that each shall bear its allotted part of the stress. The proportion of steel to concrete depends directly upon the ratio of the coefficients of elasticity of the two materials. The value of E for steel is 30,000,000 pounds per square inch, implying that a force of one pound would extend or compress a bar of steel 1 square inch in area by $\frac{1}{30,000,000}$ of its original length. The value of E for concrete may be taken as 3,000,000 pounds per square inch, so that for an equal extension of the two materials the steel will bear ten times the stress that can be borne by the concrete.

Suppose a bar of steel 1 square inch in section area to be surrounded by a ring of concrete 1 square inch in area; and suppose further that the steel be subjected to a direct pull that would occasion in the concrete the safe allowable tension stress of 50 pounds per square inch. The elongation in the concrete would then be $50 \times \frac{1}{3,000,000} = 0.0000167$ inch. Considering the bond between the steel and concrete to be perfect, the steel would suffer an equal elongation, occasioning a stress in the steel of

$$0.0000167 \times 30,000,000 = 500 \text{ pounds per square inch},$$

a result only one-thirtieth of the safe allowable tension stress for steel; consequently, the sectional area of the steel may be reduced to one-thirtieth square inch, thus raising its stress to 15,000 pounds per square inch without causing failure in the concrete.

CHAPTER X

TESTING MATERIALS

94. **Stress.** — The application of external forces to a piece of material tends to change its shape, and this tendency induces internal forces, known as stresses, which offer resistance to the change. These stresses may be of three kinds:

1. If the external force be applied at right angles to the section, and acts *away* from it, the stress is one of tension, or a tensile stress.

2. If the external force acts *toward* the section, the stress is one of compression, or a compressive stress.

3. If the external force acts *parallel* to the section, the stress is one of shear, or a shearing stress.

It is a fundamental assumption that these direct stresses are uniformly distributed over the section, so that if W denotes the external force or load, A the area of section, and S the unit stress, we must have, in the absence of rupture, $W = AS$. W is usually expressed in pounds and A in square inches, so that we shall have for the unit stress $S = \frac{W}{A}$ in pounds per square inch.

95. **Strain.** — A piece of material which is stressed by the application of external force undergoes some change in its dimensions, either lengthened or shortened, and the amount of this distortion is known as the strain due to the external force, or load.

96. **Different Kinds of Tests.** — Materials are tested for tension by pulling apart a test piece of specified dimensions; for compression, by crushing a piece of definite dimensions; for

transverse strength, by supporting a piece at two points and breaking or bending it in a testing machine by applying a load at an intermediate point; for torsion, by twisting apart a piece in a machine designed for the purpose; for direct shearing, by breaking a riveted or pin-joint connection in a machine; for impact, or shock, by letting a weight drop through a definite height, and, by its blow, develop suddenly the stress in the material.

97. Ultimate Strength. — The ultimate strength of a test piece is the load required to produce fracture, reduced to a square inch of original section; or, in other words, it is the ultimate or highest load divided by the original area. Thus, if the area of the cross section of a test piece is 0.42 square inch, and the load producing fracture is 28,400 pounds, the ultimate strength is $\frac{28,400}{0.42} = 67,620$ pounds per square inch.

98. Elastic Limit. — The elastic limit is the load per square inch of area that will just produce a permanent set in the material. Thus, in a tension test, if the cross-section area of the test piece be 0.7 square inch, and a permanent set just be produced by a load of 28,000 pounds, the elastic limit is 40,000 pounds per square inch.

99. Factor of Safety. — To insure safety in engineering construction the stresses due to the working load must not exceed the elastic limit of the material used, and to insure this provision it is customary to make the working load very considerably less than the load necessary to produce fracture. The ratio between the breaking load and the safe allowable load is the factor of safety; or it is the quotient obtained by dividing the ultimate strength by the working stress.

100. Elongation. — The increase in length of a test piece, measured just before rupture, divided by the original length of the piece, expresses the elongation.

When a load is first applied to a test piece the elongation

is nearly uniformly distributed throughout the whole length. This continues until the piece begins to contract in area near the point of final rupture, and nearly all the subsequent elongation is restricted to the immediate vicinity of this point.

In expressing the elongation of any material, the length of the test piece must be stated. If the length of the test piece is 8 inches, and an extension in length of 2 inches is noted just before rupture, the elongation is then expressed as 25 per cent in 8 inches.

101. Reduction in Area. — The reduction in area is found by dividing the difference between the original and final section areas at the point of rupture by the original area, expressing the fraction in per cent.

102. Testing Machines. — Machines devised to determine the physical properties of materials are of two general classes, — hydraulic and screw gear. With either class the load is applied to the specimen to be tested in a manner that enables it to be read instantly on some form of weighing machine.

The weighing system employed with the hydraulic machine consists usually of registering the applied load by means of a gage which records the pressure within the cylinder of an hydraulic press. Hydraulic machines, with the exception of the one devised by Emery, are lacking in sensitiveness and accuracy and are not extensively used.

Of the screw gear type of machines, those of Riehle and Olsen are very generally used. With either machine the loads are applied from some external source and are transmitted by means of spur and bevel gears to upright screws.

The Riehle machine consists of two heads, one fixed and the other movable. The upper head is fixed and is supported by two cast-iron columns which rest on the weighing table of the machine. The weighing table rests on steel knife-edges in the levers of a compound system, the last lever of which is the

weighing beam of the machine. Two upright pulling screws, which turn in long bearings and have the main gears keyed to their lower ends, pass through nuts in the movable lower head and reach nearly to the under side of the upper head. The screws raise or lower the movable head according to the direction in which they revolve.

In making a tensile test, the specimen is gripped at its end by jaws in the two heads. Power is applied from an external source, either hand or motor, in a direction to lower the movable head by the screws, thus transmitting a pull to the specimen, thence to the upper head, and then to the weighing table. The pressure on the weighing table is transmitted through the lever system to the weighing beam. On the weighing beam is a counterpoise which the operator moves along the beam to maintain the balance as the load is gradually applied, and thus has constantly under observation the magnitude of the applied load.

In making a compression test, a cylindrical cast iron block is bolted to the under side of the movable head and the specimen is placed between that and a similar block on the weighing table. As the screws lower the movable head the pressure is brought to bear on the specimen, thence through the weighing table and lever system to the weighing beam.

In making a transverse or bending test, the specimen is laid on two supports which rest on the weighing table. As the movable head is drawn down by the screws, a projection on the under side of the head bears on the middle of the specimen and the pressure is transmitted to the weighing beam, as in the other tests. The deflections of the specimen may be measured to 0.001 of an inch by attaching an instrument known as a *deflectometer*, or *transverse indicator*.

103. Forms of Test Specimens. — Test specimens are made from coupons cut from the finished product of the material to be tested, and are of prescribed form. The American Society

for Testing Materials recommends the form of specimen shown in Fig. 112 for tensile tests of metal plates, inch lengths being

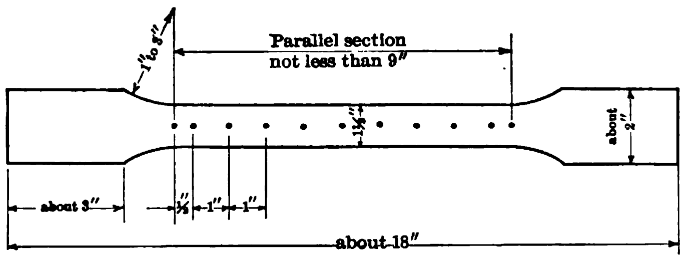


Fig. 112.

marked with center punch on the length of parallel section. Figure 113 shows the form prescribed by the Navy Department.

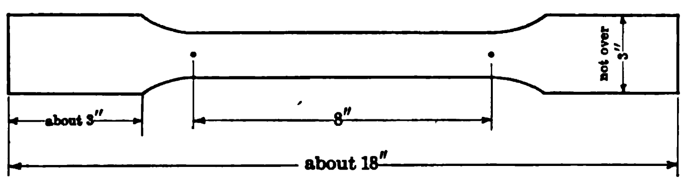


Fig. 113.

The round form of test piece shown in Fig. 114 is that used for testing manufactured products other than plates, such as shafts, axles, and beams.

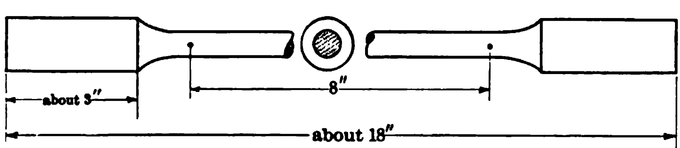


Fig. 114.

Figures 115 and 116 illustrate the cold bending and angle tests for wrought iron and steel. The material must stand these tests without sign of fracture.

The tensile test of metals is the simplest and most important, as it determines the physical properties of ultimate strength, yield point, elastic limit, and percentage of elongation.

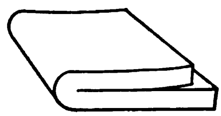


Fig. 115.

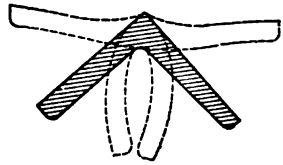


Fig. 116.

104. Ductility. — Materials such as wrought iron, mild steel, copper, and other metals which may be lengthened by the application of an external force, with a corresponding decrease in thickness or in diameter, possess the property of ductility.

105. Plasticity. — If in the process of stretching a ductile material the load be removed and none of the strain disappears, the material is said to have passed beyond the ductile and to have entered the plastic stage. The plasticity of a material is determined by the final elongation and contraction in area of the test piece. In structures subjected to live loads and shocks it is as important to know the power of the material used to resist deformation as it is to know its ultimate strength, and for that reason specifications for iron and steel usually require a certain percentage of elongation and contraction of area in a stated length of test piece.

106. Stress-strain Diagram. — If in testing a material the gradually applied loads be plotted as ordinates and the corresponding strains as abscissas, the resulting curve is known as a stress-strain diagram.

For tension tests of wrought iron and mild steel such a diagram will take the form shown in Fig. 117.

The load being gradually increased, it will be found that within a certain limit the strains, or extensions, will be directly

proportional to the augmentations in the load, and that if the stress be relieved the test piece will return to its original length. During this period the material is said to be *perfectly elastic*, and will be so represented in the diagram by the straight line OA. By continuing the gradual increase in the load a point will be reached where the proportionality between the strain and the augmentations in the load ceases, the strain increasing much more rapidly than the load; and if the stress be relieved the

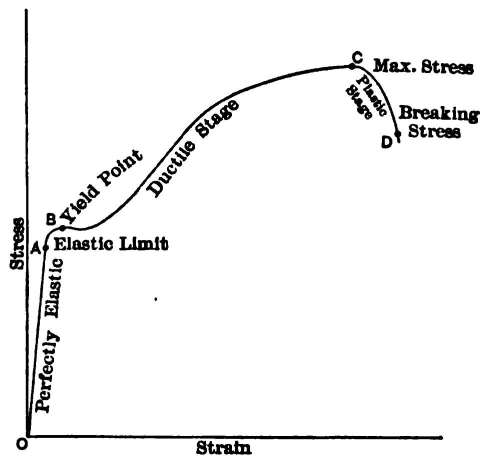


Fig. 117.

piece will not return to its original length, but will acquire a *permanent set*. The load at which this occurs is known as the *elastic limit* of the material.

A further increase in the load very soon develops a point where the extension increases very rapidly, — as much as from 10 to 15 times its previous amount, — known as the *yield point*. This rapid increase in the extension usually occasions an apparent reduction in the stress, as shown by the fall in curvature beyond the point B. For commercial purposes the yield point and the elastic limit are taken as the same point, and the ordi-

nate at *B* would represent, to the scale of the diagram, this limit in pounds per square inch of section of the material.

Passing the yield point, the strains increase much faster than the loads, but if the stress in the material be relieved a careful measurement will show the disappearance of a small portion of the extension, indicating the existence still of some elasticity and that the specimen is passing through the ductile stage.

At about the time the maximum stress is reached at *C*, the material appears to have reached the plastic state, the extension increasing, in time, without increase in load. Up to this point the strain has been evenly distributed throughout the length of the specimen, but here occurs an extension and reduction in section purely local, immediately followed by rupture at *D*.

Within the elastic limit the extension of the specimen probably would not exceed 0.001 of its length, so it is quite impossible to make direct measurements of the extensions corresponding to the augmentations in load. Some form of extensometer is used to make these measurements.

107. Extensometer. — The instrument designed to measure accurately the minute extensions within the elastic limit of a specimen during a tensile test is known as an *extensometer*. There are various forms of this instrument, but the type known as the *Riehle-Yale* is in very general use and gives dependable results.

It consists essentially of two clamps, which are fastened to the test specimen by set screws. The parallelism of the clamps is secured by a squaring gage bar which fits neatly in guides in the clamps, a set screw arrangement permitting the distance between the clamps to be varied according to the distance between the punch marks on the specimen — usually the standard distance of 8 inches — in order that the set screws of the clamps may fit exactly in the punch marks. The upper clamp has two projecting arms, 180 degrees apart, through which pass two in-

sulated bars which are connected with the lower clamp in circuit with a battery and bell. The lower clamp has projecting arms corresponding with those of the upper clamp, and through them pass two micrometer screws, each having a vertical fleet of one inch and reading to 0.0001 of an inch.

The instrument being attached to the specimen as indicated, and the squaring gage bar removed, it is ready for use. An increment of load being applied, the elongation of the specimen is measured by taking the reading, first of one of the micrometer screws and then of the other, by running them up until the contact of the point of the screw with the insulated bar completes the circuit and causes the bell to ring. Another increment of load is then added and the readings taken again, and so on until the elastic limit is reached, the average readings of the two micrometer screws for each increment of load giving the actual elongation of the specimen. The instrument is removed from the specimen after the elastic limit is passed.

108. Tensile Test of Steel. — Figure 118 is an illustration of a tensile test of steel, the scales being small in order to keep the diagram within the limits of the page.

The original dimensions of the test piece were: Length, 8 inches; diameter, 0.75 inch; area of section, 0.4418 square inch.

The final dimensions were: Length, 10.2 inches; diameter 0.4843 inch; area of section, 0.1842 square inch.

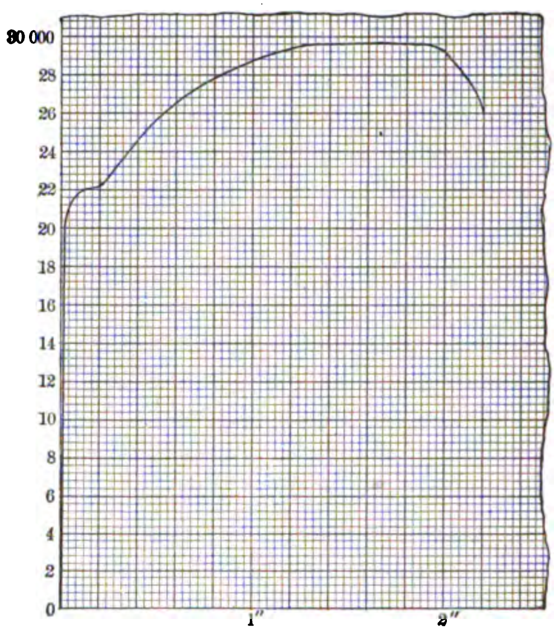
An inspection of the diagram shows the elastic limit to have been reached at about 21,000 pounds.

Elastic limit of specimen $= \dfrac{21{,}000}{0.4418} = 47{,}530$ pounds per square inch.

Ultimate stress $= \dfrac{\text{Maximum load}}{\text{Original area}} = \dfrac{29{,}700}{0.4418} = 67{,}220$ pounds per square inch.

$$\text{Extension in 8 inches} = \frac{(10.2 - 8)\,100}{8} = 27.5 \text{ per cent.}$$

$$\text{Contraction of area} = \frac{(0.4418 - 0.1842)\,100}{0.4418} = 58.31 \text{ per cent.}$$



Scales: Loads, 0.2″=2000 lbs.; Extensions, full size.

Fig. 118.

Loads.	Extensions.	Remarks.
2,000	0.0012	
4,000	0.0024	
6,000	0.0034	
8,000	0.0047	
10,000	0.0060	
12,000	0.0072	
14,000	0.0083	
16,000	0.0095	
18,000	0.0106	
20,000	0.0118	
22,000	0.0220	Yield point
26,000	0.5300	
29,680	1.9200	Maximum stress
25,820	2.2000	Breaking stress

The load was gradually applied with the uniform augmentation of 2000 pounds, and the data of the test was tabulated as shown in the table.

To find the modulus of elasticity we proceed as follows:

The sum of the extensions up to and including the 18,000 pound load — a point well within the limit of elasticity — is 0.0533 inch, and the sum of the loads is 90,000 pounds. The mean extension for 2000 pounds is, therefore, $\frac{0.0533}{45} = 0.0012$ inch.

By Art. 31, p. 63, we have $E = \frac{SL}{y}$, in which L is the original length and S the load producing the extension y. Here $L = 8$ inches, $S = 2000$ pounds, and $y = 0.0012$ inch.

Hence, $$E = \frac{8 \times 2000}{0.4418 \times 0.0012} = 30{,}180{,}000 \text{ lbs. per sq. inch.}$$

109. Compression Tests. — In testing materials for compression, the specimens are not longer than 1.5 to 3 times the diameter. If the specimens are long the failure under compression will be by buckling or bending, and for intermediate lengths partly by crushing and partly by bending.

AVERAGE PHYSICAL PROPERTIES OF MATERIALS.

Material.	Pounds per square inch.							Pounds.
	Elastic limit.			Ultimate strength.			Modulus of elasticity.	Weight per cubic foot.
	Tension.	Compression.	Shearing.	Tension.	Compression.	Shearing.		
Hard steel	60,000	....	....	100,000	....	....	31,000,000	490
Mild steel	36,000	33,000	24,000	60,000	60,000	50,000	30,000,000	490
Cast steel	25,000	....	....	50,000	....	....	27,000,000	490
Tool steel, unhardened	90,000	....	....	120,000	....	....	31,000,000	490
Tool steel, hardened	170,000	....	....	170,000	....	....	33,000,000	490
Wrought iron	28,000	....	25,000	50,000	55,000	40,000	25,000,000	480
Cast iron	6,000	20,000	....	25,000	90,000	20,000	15,000,000	450
Copper	7,000	24,000	....	30,000	49,000	....	15,000,000	550
Concrete	....	1,000	....	300	2,500	1,400	3,000,000	150
Timber	3,000	....	....	10,000	8,000	600	1,500,000	40

PART II

THE ELEMENTS OF POWER TRANSMISSION

CHAPTER I

TRANSMISSION OF POWER BY BELTS AND ROPES

1. **Flat Belt Gearing.** — The transmission of power by means of belts running over pulleys is an important and familiar mechanical contrivance. It is practically noiseless, and may be used for transmitting power through a distance as great as 30 feet without intervening support. For greater distances idle or binder pulleys are generally used to tighten the slack side of the belt.

The principal disadvantage of belt drives is that due to the slip occasioned by the freedom of the belt to slip over the pulley, rendering the transmission not so positive as that through the medium of gear wheels; but this disadvantage becomes an advantage in preventing shocks in cases where mechanisms at rest are suddenly thrown into gear.

2. **Materials for Flat Belts.** — The most common material for flat belting is leather, though cotton and India rubber are not infrequently used. The best leather belting is made of oxhide, the strips of the hide being tapered at the ends and cemented together under great pressure, and then laced or riveted with copper rivets and washers. The thickness of the *single-ply* belt varies from $\frac{3}{16}$ to $\frac{5}{16}$ inch. If a greater thick-

ness is required, two or three strips are cemented together under pressure, thus producing *two-ply* and *three-ply* belting in varying thickness up to ½ inch. The average weight of leather belting is 0.036 pound to the cubic inch.

Cotton belts are usually made by stitching together canvas or ducking, but are sometimes woven solid. Such belts are cheaper and equally as strong as those made of leather, but the coefficient of friction is rather low unless the material is properly *sized*. The weight of cotton belting varies with the material, but the average of a cubic inch may be taken as 0.034 pound.

Rubber belting is produced by treating canvas with a composition of rubber in such manner as to fill all its interstices. It is then wrapped in rubber and vulcanized under heat and pressure. It is adaptable to use in damp places, since the material is not affected by moisture.

The weight of rubber belting is about 0.044 pound per cubic inch.

3. **Strength of Leather Belting.** — The ultimate strength of leather used for belting varies from 3600 to 5000 pounds per square inch of section and the strength of the laced joint may be taken as one-third that of the solid leather. Taking a factor of safety of 5, the safe working stress of a laced joint varies from 240 to 330 pounds per square inch of section for single-ply belting, and from 500 to 600 pounds per square inch of section for double-ply. The width of the belt must be made sufficient to withstand the tension. The safe tension in pounds per inch of width of single-ply belting ranges from 50 to 80 according to the thickness and the safe stress of section of the material.

4. **Coefficient of Friction of Leather Belting.** — The coefficient of friction of leather belting running over smooth iron pulleys is a very important but exceedingly variable quantity, ranging from 0.22 to 0.35. With wooden pulleys having varnished faces the coefficient of friction is somewhat greater.

5. Velocity of Leather Belting. — The velocity of leather belting varies from 2000 to 6000 feet per minute, but the most economical speed is from 4000 to 5000 feet. At higher speeds the effect of centrifugal action is excessive and the life of the belt shortened.

6. Velocity Ratio in Flat-Belt Transmission. — If there is no slipping of the belt on the pulleys when transmitting motion from one pulley to another, the outer surface of the rim of each pulley will have the same velocity as the belt; so if we denote by D_1 and D_2 the diameters of the driver and follower pulleys respectively, and by N_1 and N_2 their revolutions per minute, we shall have

$$\text{Speed of outer surface of driver rim} = \pi D_1 N_1,$$

and $$\text{Speed of outer surface of follower rim} = \pi D_2 N_2,$$

and since each of these is equal to the speed of the belt, we have

$$\pi D_1 N_1 = \pi D_2 N_2, \quad \text{whence} \quad \frac{N_1}{N_2} = \frac{D_2}{D_1}.$$

That is, the velocities of the pulleys are inversely as their diameters.

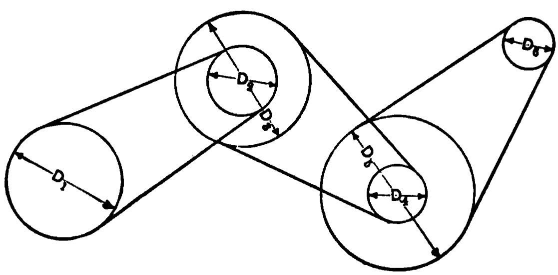


Fig. 1.

When motion is transmitted from one shaft to another by belting, one or more shafts intervening, as in Fig. 1, we shall have

$$\frac{N_1}{N_2} = \frac{D_2}{D_1}, \quad \frac{N_3}{N_4} = \frac{D_4}{D_3}, \quad \text{and} \quad \frac{N_5}{N_6} = \frac{D_6}{D_5}.$$

Taking the product of these equations, member by member, and remembering that $N_2 = N_3$, and $N_4 = N_5$, we have

$$\frac{N_1}{N_6} = \frac{D_2}{D_1} \times \frac{D_4}{D_3} \times \frac{D_6}{D_5}.$$

That is, the ratio of the speed of the first pulley to the speed of the last pulley is equal to the continued product of the ratios of the diameters of each pair of pulleys taken in order.

The results just obtained are true only for very thin belts, or for belts whose thickness is so small in comparison with the diameters of the pulleys as to be negligible. The thickness of belts ordinarily used may have an appreciable effect on the velocity ratio of two pulleys. While the belt is in contact with the pulley, its inner surface is in compression and its outer surface in tension, but the neutral surface midway between is of constant length. It follows that the velocity of the surface of the belt in contact with the pulley is less than the velocity of the neutral surface, and that the true velocity ratio of two pulleys is obtained by increasing the diameters of the pulleys by an amount equal to the thickness of the belt.

7. **Slip of Flat Belts.** — Another error in the velocity ratio of belt-driven pulleys is due to *slip*. The driving side of a belt is necessarily stretched more than the slack side, and in consequence the driving pulley receives a greater length of belt than it gives off to the follower pulley; therefore the speed of the driving side is a trifle greater than that of the slack side. But the speed of the rim of a pulley is the same as that of the belt it receives; therefore the speed of the rim of the driver will be somewhat greater than that of the rim of the follower, and, as a consequence, the speed of the follower will be less than that given by the ratio equation $\frac{N_1}{N_2} = \frac{D_2}{D_1}$. This difference between the speeds of the rims of the driver and the follower is known as *slip*, and amounts to about 2 per cent.

Example I. — The pulley on the shaft of a steam engine is 4 feet in diameter, from which a belt passes to a pulley 2 feet in diameter on a shaft in a room above; a belt passes from another 2-foot pulley on this shaft to one of 10 inches in diameter on a third shaft; an 18-inch pulley on the third shaft is belted to a 6-inch pulley on the spindle of a dynamo. Find the speed of the dynamo when the engine is making 160 revolutions per minute.

Solution. — $$\frac{\text{Speed of engine}}{\text{Speed of dynamo}} = \frac{24}{48} \times \frac{10}{24} \times \frac{6}{18} = \frac{5}{72},$$
whence
$$\text{Speed of dynamo} = \frac{160 \times 72}{5} = 2304 \text{ revolutions per minute.}$$

If the thickness of the belts in this example were 0.25 inch and the slip 2 per cent, we should have

$$\text{Speed of dynamo} = \frac{48.25}{24.25} \times \frac{24.25}{10.25} \times \frac{18.25}{6.25} \times 160 \times 0.98 = 2155$$

revolutions per minute, showing an error of 6.9 per cent in computing the speed of the dynamo by the usual method.

When great accuracy is required, the errors due to slip and to the failure to include the thickness of the belt in the pulley diameters should be corrected.

8. Tensions in Flat Belts. — When a belt is fitted to two pulleys it is strained over them while at rest with a tension T_0, which is uniform throughout the belt. When motion ensues, the driving side of the belt stretches and its tension T_1 increases. At the same time the slack side of the belt is shortened and its tension T_2 decreases. The theory of belting rests on the assumption of perfect elasticity in the belts, so that the lengthening of the driving side of the belt must equal in amount the shortening of the slack side, and the average tension in the belt remains constant. That is, we should have

$$T_1 + T_2 = 2\,T_0.$$

Such is not the case, however, as the materials used for belting are far from being perfectly elastic; but the error involved is not of such consequence as to destroy the theory when applied to the conditions under which belts are ordinarily used.

9. Frictional Resistance between a Flat Belt and a Pulley. — Let the two pulleys of Fig. 2 be joined with an endless belt. If a force tends to turn the driving pulley D in the direction indicated by the arrow, the lower part of the belt will be stretched

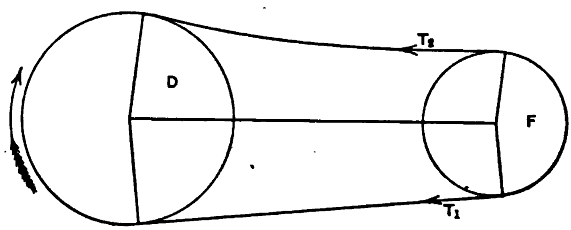


Fig. 2.

so that its tension T_1 will be increased, and the tension T_2 in the upper part of the belt will be decreased an equal amount. Each element of the belt in contact with the pulley assists the action of the tension in the slack side of the belt in resisting the tension in the tight side, and therefore the tension in that part of the belt in contact with the pulley varies at every point. When the difference, $T_1 - T_2$, becomes sufficient to overcome the resistance to motion in the driven pulley F, its rotation begins. The difference in tensions in the two sides of the belt is the amount of friction between the belt and the pulley, and is the measure of the driving force. If the tensions were equal, their moments about the center of the driven pulley would be equal and there would be no rotation, since there would be equal turning tendencies in opposite directions.

Let T_1 and T_2, Fig. 3, denote respectively the tensions in the tight and slack parts of the belt that are not in contact with

the pulley. Suppose $d\alpha$ to be a very small part of the central angle α subtended by the arc of contact of the belt.

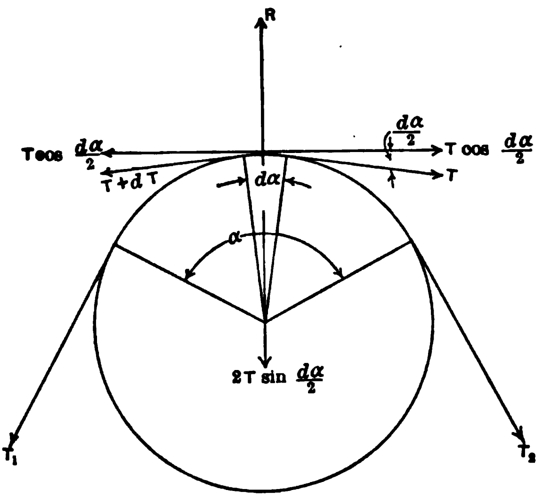


Fig. 3.

Considering the equilibrium of the very small arc subtending $d\alpha$, we will assume the tensions at its extremities to be T and $T + dT$. If the resulting reaction between the belt and the pulley rim due to these tensions be denoted by R, we shall have the static equation

$$R = 2\ T \sin \frac{d\alpha}{2} = Td\alpha,$$

since $d\alpha$ is very small and $\sin \frac{d\alpha}{2}$ and $\frac{d\alpha}{2}$ are very approximately the same.

As slipping is about to take place, dT is the measure of the friction over the small arc considered.

Then $$dT = \mu R = \mu T d\alpha,$$

in which μ is the coefficient of friction.

For the small arc considered, $\frac{dT}{T} = \mu d\alpha$, and for the whole arc of contact we have

$$\int_{T_2}^{T_1} \frac{dT}{T} = \int_0^{\alpha} \mu d\alpha,$$

from which

$$\log T_1 - \log T_2 = \mu\alpha, \quad \text{or} \quad \log \frac{T_1}{T_2} = \mu\alpha,$$

whence $$\frac{T_1}{T_2} = e^{\mu\alpha},$$

in which $e = 2.718$, the base of the Naperian system of logarithms, and α is expressed in circular measure. If α is given in degrees it may be converted into radians by multiplying by $\frac{\pi}{180}$.

Hence, it is seen that the ratio of the tensions depends only upon the coefficient of friction and the angle at the center subtended by the arc of contact of the belt, and is independent of the diameter of the pulley.

A convenient expression for the ratio of the tensions is

$$\log_{10} \frac{T_1}{T_2} = \mu\alpha \log_{10} 2.718 = 0.4343\,\mu\alpha.$$

The result just obtained is applicable to a rope making several turns about a post, and explains how it is possible for one man to check the headway of a ship when docking by taking several turns of a hawser about a post on the dock and pulling the slack end with a comparatively small force.

Example II. — A hawser from a ship makes three turns about a post and is pulled with a force of 50 pounds. The coefficient of friction being 0.35, what force is exerted in checking the headway of the ship?

Solution. —

Here $T_2 = 50, \quad \alpha = 2\pi$ radians, and $\mu = 0.35$.

Then $$\log \frac{T_1}{T_2} = 0.4343 \times 0.35 \times 6\pi = 2.86638,$$

whence

$$\frac{T_1}{T_2} = 735.15, \text{ and } T_1 = 735.15 \times 50 = 36{,}757 \text{ pounds.}$$

10. Transmission of Power by Flat Belts. — In a belt connection between two pulleys the arc of contact to be considered is that of the smaller pulley, and if the connection is by open belt the arc of contact of the smaller pulley will be less than 180 degrees, but with a crossed belt the arc of contact will be greater than 180 degrees. It follows that with the same conditions, a greater power may be transmitted with the crossed belt. The crossed belt, however, is subject to great wear, and generally its use is restricted to cases where it is desired to have the pulleys turn in opposite directions.

The friction between a belt and pulley limits the power that can be transmitted. When overloaded, a belt will slip rather than break; therefore at the point of slipping the driving force is $T_1 - T_2$, and we shall have

$$\text{H.P. transmitted} = \frac{(T_1 - T_2)V}{33{,}000} = \frac{(T_1 - T_2)v}{550},$$

according as to whether the velocity is expressed in feet per minute or in feet per second.

The ratio of the tensions is given by

$$\log \frac{T_1}{T_2} = 0.4343\ \mu\alpha.$$

Example III. — An open belt 3 inches wide connects a pulley 5 feet in diameter with one 18 inches in diameter, the distance between the pulley centers being 12 feet. The larger pulley makes 200 revolutions per minute, the coefficient of friction is 0.35, and the greatest tension in the belt must not exceed 80 pounds per inch of width. Find the horse power transmitted.

Solution.—

$$\text{Velocity of belt in feet per second} = \frac{5\pi \times 200}{60} = 52.36.$$

$$T_1 = 80 \times 3 = 240 \text{ pounds.}$$

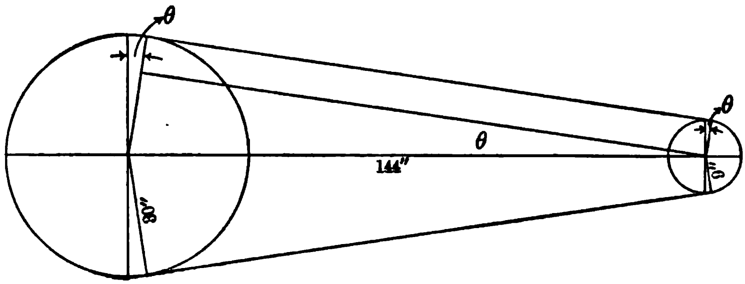


Fig. 4.

Referring to Fig. 4,

$$\sin \theta = \frac{30 - 9}{144} = 0.1458, \quad \text{whence } \theta = 8^\circ\ 23'.$$

$$\text{Arc of contact of smaller pulley} = [180^\circ - 2\,(8^\circ\ 23')]\frac{\pi}{180}$$
$$= 2.85 \text{ radians.}$$

$$\log \frac{T_1}{T_2} = 0.4343\mu\alpha = 0.4343 \times 0.35 \times 2.85 = 0.43320,$$

therefore

$$\frac{T_1}{T_2} = 2.71, \quad \text{whence} \quad T_2 = \frac{T_1}{2.71} = \frac{240}{2.71} = 88.56 \text{ pounds.}$$

Then

$$\text{H.P.} = \frac{(T_1 - T_2)\,v}{550} = \frac{(240 - 88.56)\,52.36}{550} = 14.42.$$

11. Centrifugal Action in Belts. — When belts are run at high speed the tensions are greater than those due to the power transmitted on account of the centrifugal action in that part of the belt in contact with the pulley. This centrifugal action also diminishes the normal pressure of the belt on the pulley rim, and therefore decreases the frictional resistance. For

speeds beyond 3500 feet per minute the stress due to centrifugal action increases rapidly and should be taken into account.

Let T denote the tension due to centrifugal action in the belt of Fig. 5, the arc of contact being α.

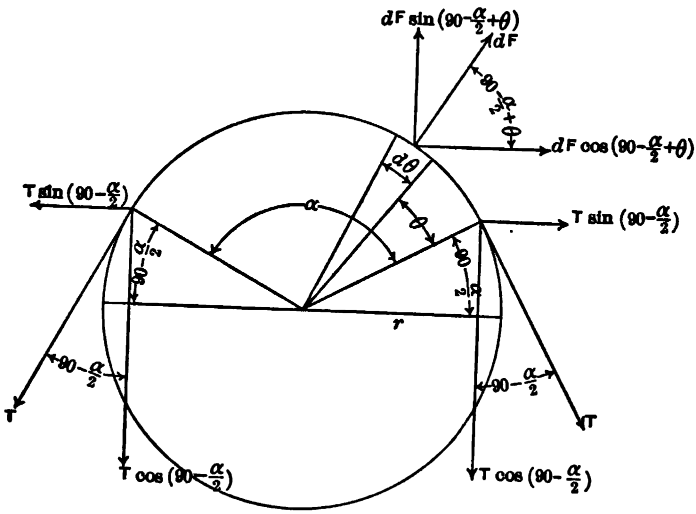


Fig. 5.

Consider the very small arc of the pulley subtending the angle $d\theta$, and let dF denote the centrifugal force set up in the part of the belt in contact with this arc.

We have the static equation

$$2\,T\cos\left(90^\circ - \frac{\alpha}{2}\right) = \int dF \sin\left(90^\circ - \frac{\alpha}{2} + \theta\right),$$

or

$$2\,T\sin\frac{\alpha}{2} = \int dF\cos\left(\frac{\alpha}{2} - \theta\right).$$

$$dF = \frac{dmv^2}{r}.$$

If W_1 denotes the weight of the belt in pounds per linear foot,

then
$$dm = \frac{W_1}{g} \cdot r d\theta,$$

and
$$dF = \frac{W_1 r d\theta v^2}{gr} = \frac{W_1 v^2 d\theta}{g}.$$

Then

$$2\,T \sin\frac{\alpha}{2} = \frac{W_1 v^2}{g}\int_0^{\alpha} \cos\left(\frac{\alpha}{2} - \theta\right) d\theta = \frac{W_1 v^2}{g}\left[-\sin\left(\frac{\alpha}{2} - \theta\right)\right]_0^{\alpha}$$

$$= \frac{W_1 v^2}{g} \cdot 2 \sin\frac{\alpha}{2},$$

whence
$$2\,T = \frac{2\,W_1 v^2}{g} = \frac{2\,r W_1}{g} \cdot \frac{v^2}{r} = \frac{W v^2}{gr},$$

and
$$T = \frac{W v^2}{2\,gr},$$

in which $W = 2rW_1$ is the weight of a portion of the belt whose length equals the diameter of the pulley; v the velocity in feet per second; g the acceleration of gravity, 32.2 feet per second per second; and r the radius of the pulley in feet.

Example IV. — Find the width and thickness of belt necessary to transmit 10 horse power to a 15-inch pulley so that the greatest tension may not exceed 60 pounds per inch width of belt when the pulley makes 1200 revolutions per minute, the weight of the belt per square foot being 1.5 pounds, the coefficient of friction 0.25, and the arc of contact of the belt 165 degrees. The weight of a cubic inch of leather is 0.036 pound.

Solution: —

$$v = \frac{\pi \times 15 \times 1200}{12 \times 60} = 78.54 \text{ feet per second.}$$

$$\text{H.P. per second} = \frac{(T_1 - T_2)\,v}{550},$$

whence $$T_1 - T_2 = \frac{550 \times 10}{78.54} = 70 \text{ pounds.}$$

$$\alpha = \text{arc of contact} = \frac{165^\circ \times \pi}{180} = 2.88 \text{ radians.}$$

We have, $$\log \frac{T_1}{T_2} = 0.4343\,\mu\alpha = 0.4343 \times 0.25 \times 2.88 = 0.31270,$$

whence $$\frac{T_1}{T_2} = 2.05.$$

Then $2.05\,T_2 - T_2 = 70$, whence $T_2 = 66.67$ pounds,

and $$T_1 = 2.05 \times 66.67 = 136.67 \text{ pounds.}$$

The effect of centrifugal action increases the tension by $\frac{Wv^2}{2\,gr}$ pounds. If w denotes the width of the belt in inches, then W = weight of a portion of the belt whose length equals the diameter of the pulley $= 2\,rW_1$ = weight of $\frac{2\,rw}{12}$ square feet of the belting $= \frac{2\,rw \times 1.5}{12}$ pounds.

Then $$\frac{Wv^2}{2\,gr} = \frac{2\,rw \times 1.5 \times (78.54)^2}{2\,gr \times 12} = 23.95\,w,$$

and $$T_1 = 136.67 + 23.95\,w.$$

But from the conditions of the problem the tension must not exceed $60\,w$;

hence $60\,w = 136.67 + 23.95\,w$, whence $w = 3.79$ inches.

If t denotes the thickness of the belt in inches, then $144\,t$ is the volume in cubic inches of 1 square foot of the belting, and since 1 cubic inch of leather weighs 0.036 pound, we have

$$144\,t \times 0.036 = 1.5, \text{ whence } t = 0.29 \text{ inch.}$$

12. Flat Belt Connections between Non-parallel Shafts. — Two non-parallel and non-intersecting shafts may be connected by a belt, provided the pulleys are so placed that the point at which the belt leaves either pulley lies in the plane of the other

pulley, and provided further that the belt runs only in one direction.

As an illustration, Fig. 6 shows the arrangement of a belt making a *quarter turn*, the shafts being at right angles. It will

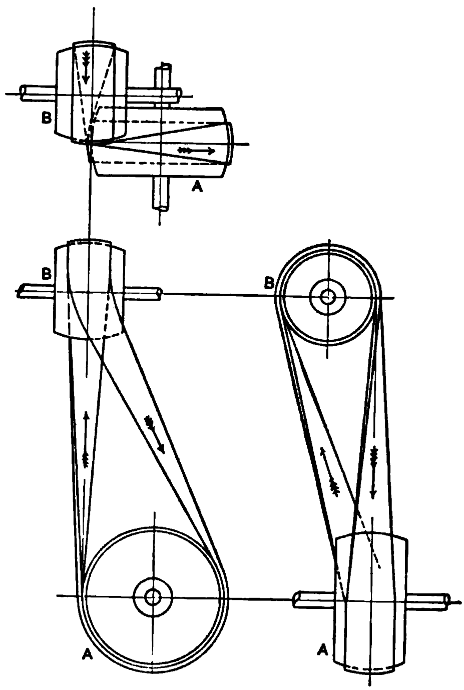


Fig. 6.

be noted that the belt in passing from pulley *A* approaches pulley *B* in a direction at right angles to the axis of pulley *B*, and in passing from pulley *B* it approaches pulley *A* in a direction at right angles to the axis of pulley *A*. It will be noted, too, that a plane that is tangent to the rim of one pulley and perpendicular to the axis of the other pulley cuts the other pulley in the middle of its rim.

Non-parallel shafts, whether they intersect or not, may be connected by a belt to run in either direction by means of inter-

mediate guide pulleys placed on a spindle whose axis is the intersection of the middle planes of the principal pulleys.

13. **Rope Gearing.** — Ropes running over pulleys having *V*-shaped grooves in their rims are used in preference to belts in cases where much power is to be transmitted. The horizontal distance between the pulley shafts may be as great as 90 feet, the ropes between the pulleys hanging in catenary curves. The materials of which non-metallic ropes are usually made are cotton and manila hemp, the former being the better on account of its greater flexibility and higher coefficient of friction.

14. **Systems of Rope Gearing.** — There are two systems of rope transmission, known as the *multiple* and the *continuous.*

In the multiple system there is an endless rope for each groove of the pulley system, whereas in the continuous system there is but one endless rope which, when it leaves the first groove of the driving pulley, enters the first groove of the follower pulley. Leaving the first groove of the follower, the rope enters the second groove of the driver and leaves that groove to enter the second groove of the follower, thence to the third groove of the driver, and so on until leaving the last groove of the follower, when it is directed by a guide pulley to enter the first groove of the driver.

The continuous system is particularly adaptable to cases where the distance between the driving and the driven shafts is short, and it has the distinct advantage of having but one splice. A marked disadvantage lies in the fact that a breakage of the rope disables the whole system, which would not be occasioned by the breakage of any one of the ropes of a multiple system.

The ropes most commonly used vary in size from 1 inch to 2 inches in diameter, though smaller sizes are used for general transmission in manufacturing establishments.

Each turn in the coil of the rope in the continuous system

has the same tension, so that the driving force is the same as that of a multiple system having the same number of separate ropes as the continuous system has turns in its coil.

15. Strength, Weight, and Velocity of Rope Belts. — The breaking strength of ropes varies from 7000 to 12,000 pounds per square inch of section, but to insure durability it is good practice to limit the working stress of a rope to about 150 pounds per square inch, indicating a factor of safety of 60. The working stress of a rope may be taken as $120d^2$, in which d is the diameter of the rope in inches.

The weights per linear foot of manila and cotton ropes are given very approximately by $0.3d^2$ and $0.28d^2$ respectively, d being the diameter in inches.

The velocity of transmission rope varies from 3000 to 6000 feet per minute, the most efficient speed being 4700 feet.

16. Frictional Resistance between Rope and Grooved Pulley. — The V-shaped grooves in the rims of pulleys used for non-metallic rope drives are so dimensioned that the rope presses only on the sides of the groove and not on the bottom, thus securing a greater resistance to slipping.

Figure 7 represents a pulley of a rope drive and a cross section of the groove and rope. Let α denote the arc of contact of the rope and 2θ the angle of the groove. Let R denote the resultant central force due to the normal pressures P and P of the rope on the sides of the groove. Consider the small length of arc of the rope subtending the angle $d\alpha$, and let Q denote the resisting pressure at each side of the groove in contact with the rope. We have the static equation

$$2Q \sin \theta = R = 2T \sin \frac{d\alpha}{2} = Td\alpha. \quad \text{(See Art. 9, p. 224.)}$$

The measure of the friction over the small arc considered is

$$dT = \mu \times 2Q = \mu \times Td\alpha \times \operatorname{cosec} \theta,$$

whence $$\frac{dT}{T} = \mu \operatorname{cosec} \theta d\alpha.$$

Then $$\int_{T_2}^{T_1} \frac{dT}{T} = \mu \operatorname{cosec} \theta \int_0^{\alpha} d\alpha,$$

whence $$\log \frac{T_1}{T_2} = \mu\alpha \operatorname{cosec} \theta,$$

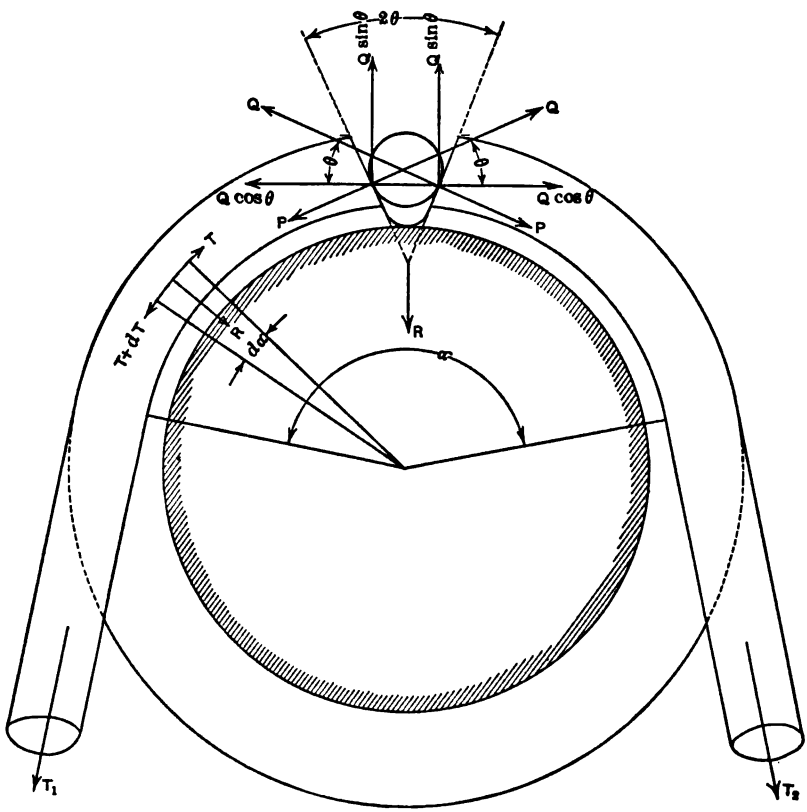


Fig. 7.

in which θ is half the angle between the sides of the groove. As the groove angle is usually about 45 degrees, the value of θ is 22.5 degrees, and cosec θ = 2.6.

Then $$\log \frac{T_1}{T_2} = 2.6\,\mu\alpha,$$

in which μ is the coefficient of friction and α the arc of contact of the smaller pulley.

Comparing this result with that obtained in Art. 9, p. 224, we see that the logarithm of the tension ratio of the grooved pulley is 2.6 times as great as that obtained for the flat pulley.

The coefficient of friction for well lubricated manila ropes lies between 0.12 and 0.15, and for cotton ropes between 0.18 and 0.28.

The ratio $\frac{T_1}{T_2}$ for ropes is much greater than that for flat belts owing to the wedging action in the grooves.

The effect of centrifugal action in ropes is appreciable and may be determined in the same manner as for flat belts.

17. **Transmission of Power by Ropes.** — As in the case of flat belts, the driving force in rope transmission is $T_1 - T_2$, and the equation for the tension ratio of Art. 9, p. 224, is applicable in determining the power transmitted by ropes if we substitute μ cosec θ for μ, θ being half the angle of the groove. We shall then have

$$\text{H.P. } per\ rope = \frac{(T_1 - T_2)\,v}{550}, \quad \text{and } \log \frac{T_1}{T_2} = \mu\alpha \operatorname{cosec} \theta = 2.6\,\mu\alpha,$$

when $2\,\theta = 45$ degrees, which is commonly the case.

Example. — How many ropes of 1.125 inches in diameter will be required to transmit 200 H.P. from a pulley 6 feet in diameter making 250 revolutions per minute, the coefficient of friction being 0.15 and the arc of contact 150 degrees?

Solution. —

$$T_1 = 120d^2 = 120 \times (1.125)^2 = 151.88 \text{ pounds.}$$

$$\text{Arc of contact} = \frac{150\,\pi}{180} = 2.618 \text{ radians.}$$

$$\log \frac{T_1}{T_2} = 2.6\,\mu\alpha = 2.6 \times 0.15 \times 2.618 = 1.02102$$

therefore $\dfrac{T_1}{T_2} = 10.5$ and $T_2 = \dfrac{151.88}{10.5} = 14.46$ pounds.

$$\text{Velocity of ropes} = \frac{\pi \times 6 \times 250}{60} = 78.54 \text{ feet per second.}$$

$$\text{H.P. per rope} = \frac{(T_1 - T_2)\,v}{550} = \frac{(151.88 - 14.46)78.54}{550} = 19.62.$$

$$\text{Number of ropes required} = \frac{200}{19.62} = 10.2, \text{ say } 10.$$

18. Telodynamic Transmission. — The method of transmitting power over long distances by means of wire ropes and pulleys has been given the name of *telodynamic transmission.* Wire ropes are enormously stronger than ropes made of manila and cotton, and when run at high velocities are very efficient in the transmission of power. Their excessive wear and great cost of replacement have proven to be such serious disadvantages as to restrict the employment of a system of transmission otherwise admirable.

The wire ropes commonly used have six strands, each of which contains six wires of diameters varying from 0.02 to 0.083 inch. The strands are wound around a central core of hemp, and then the six strands are twisted around the central core of the rope, also made of hemp.

Wire ropes are made of iron and steel, the latter material being the better. The working stress of wire rope has been fixed at 25,600 pounds per square inch of section, and the weight per foot run may be taken as $1.34d^2$ pounds, in which d is the diameter of the rope in inches.

Owing to the excessive wear which would be occasioned by wedging wire ropes in the pulley grooves, as is done with hemp and cotton ropes, the grooves are so made as to permit the ropes to ride on the bottom.

The equations found in Arts. 9 and 11 are applicable to wire ropes, the coefficient of friction being about 0.24.

PROBLEMS

1. Two pulleys, 15 inches and 37.5 inches in diameter, are connected by a belt. If the 15-inch pulley makes 500 revolutions in 10 minutes, how many turns will be made by the other pulley in 25 minutes? *Ans.* 500.

2. Sketch an arrangement of four pulleys with belts for driving a fan at 1500 revolutions per minute from a shaft making 200 revolutions per minute, giving the diameters of the pulleys to be used.

3. An engine shaft, making n revolutions per minute, carries a 56-inch pulley which drives, by means of a belt, a 36-inch pulley on the line shaft. The line shaft carries another pulley, 42 inches in diameter, which is belted to a 24-inch pulley on a counter shaft. Another pulley on the counter shaft is 48 inches in diameter and is belted to a 14-inch pulley on the spindle of a dynamo. Find the number of revolutions made in a minute by the dynamo spindle. *Ans.* $9.33n$.

4. The flywheel of an engine is 28 inches in diameter and is belted to a pulley 20 inches in diameter on another shaft. A 20-inch pulley on the second shaft is belted to a 10-inch pulley on a third shaft, which carries an 18-inch pulley, which, in turn, is belted to a 6-inch pulley on the spindle of a dynamo. Find the speed of the dynamo when the engine is making 90 revolutions per minute. If the belt thickness of three-sixteenths inch be considered, find the per cent of loss in the speed of the dynamo.

Ans. 756 revolutions; 3.18 per cent.

5. The shaft of a high-speed engine, which is making 300 revolutions per minute, carries a 14-inch pulley, over which passes a belt to a 20-inch pulley on another shaft. A 10-inch pulley on this shaft drives a 20-inch pulley on a third shaft carrying a 6-inch pulley which is to be belted to the spindle of a machine so that the revolutions of the spindle may be 45 per minute. Find the diameter of the pulley on the spindle.

Ans. 14 inches.

6. A weight of 8000 pounds is suspended from one end of a rope. How many turns of the rope must be taken around a circular beam fixed horizontally in order that a man, who can pull with a force of 250 pounds, may keep the rope from slipping, supposing the coefficient of friction to be 0.2?

Ans. 2.75.

7. By taking 3 turns of a rope about a post, and holding back with a force of 180 pounds, a man just keeps the rope from slipping. The coefficient of friction being 0.2, find the weight supported at the other end of the rope. *Ans.* 7810 pounds.

8. The pulley on an engine shaft is 5 feet in diameter and makes 100 revolutions per minute. The motion is transmitted from this pulley to the main shaft by a belt running on a pulley, the difference in tensions

between the tight and slack sides of the belt being 115 pounds. What is the work done per minute in overcoming the resistance to motion of the main shaft? *Ans.* 180,714 foot pounds.

9. A belt having a linear velocity of 350 feet per minute transmits 5 H.P. to a pulley. Find the tension in the driving side, supposing it to be double that in the slack side. *Ans.* 943 pounds.

10. A pulley 3 feet 6 inches in diameter, and making 150 revolutions per minute, drives by means of a belt a machine which absorbs 7 H.P. What must be the width of the belt so that its greatest tension shall be 70 pounds per inch of width, it being assumed that the tension in the driving side is twice that in the slack side? *Ans.* 4 inches.

11. Find the horse power that may be transmitted by a belt 8 inches wide and passing over a 20-inch pulley on the shaft of an engine which makes 350 revolutions per minute. The angle at the center subtended by the arc of contact of the belt is 160 degrees, and the coefficient of friction is 0.4. The tension in the driving side may be taken as 80 pounds per inch of width, and the stress per square inch of belt section must not exceed 300 pounds. Find also the thickness of the belt. *Ans.* 23.94 H.P.; 0.27 inch.

12. A belt is to transmit 2 H.P. from a pulley 12 inches in diameter on a shaft making 160 revolutions per minute. Find: (1) the tensions in the driving and in the slack sides of the belt when the arc of contact is 180 degrees and the coefficient of friction is 0.3. (2) The width of the belt when the thickness is 0.25 inch, and the safe working stress 320 pounds per square inch of belt section.

Ans. 215.15 pounds; 83.85 pounds; 2.69 inches.

13. What horse power will a belt 8 inches wide transmit over an 18-inch pulley making 300 revolutions per minute, the weight of 1 square foot of the belt being 1.29 pounds, the coefficient of friction 0.3, the arc of contact of the belt 160 degrees, and the tension per inch of width of the belt not to exceed 80 pounds? Take the weight of a foot of belting 1 square inch in section as 0.43 pound. *Ans.* 15.8.

14. A leather belt is required to transmit 2 H.P. from a shaft running at 80 revolutions per minute to a shaft running at 160 revolutions per minute. Find the stresses in the belt, assuming that the smaller pulley is 12 inches in diameter, and that the ratio of the tensions in the tight and slack sides of the belt is 2.25 : 1. Find also the width of belt, taking the working stress at 100 pounds per inch of width.

Ans. 236.3 pounds; 105 pounds; 2.36 inches.

15. Find the width of belt necessary to transmit 10 H.P. to a pulley 12 inches in diameter so that the greatest tension may not exceed 40 pounds per inch of width when the pulley makes 1500 revolutions per minute. The

weight of the belt per square foot is 1.5 pounds, the coefficient of friction 0.25, and the arc of contact 180 degrees. The weight of a cubic inch of leather may be taken as 0.036 pound. (The effect of the thickness of the belt and of centrifugal action must be taken into consideration.)

Ans. 8.4 inches.

16. An 8-inch belt traveling over a 30-inch pulley making 174 revolutions per minute transmits 18 H.P. The ratio of the tensions in the tight and slack sides of the belt is 3.06 : 1, the arc of contact 160 degrees, and the maximum tension allowed per inch width of belt 80 pounds. Taking the velocity of the neutral surface of the belt as the true velocity, it is required to find: (1) The coefficient of friction. (2) The thickness of the belt.

Ans. 0.4; 0.25 inch.

17. A driving shaft, making 100 revolutions per minute, carries a pulley 22 inches in diameter, from which a belt communicates motion to a 12-inch pulley on a countershaft. On the countershaft is also a cone pulley having steps of 8, 6, and 4 inches in diameter, which gives motion to another cone pulley, of equal steps, on a lathe spindle. Sketch the arrangement in side and end elevations, and find the greatest and least speeds at which the lathe spindle can revolve. *Ans.* 366.66; 91.66.

18. Determine the horse power that may be transmitted by a belt 6 inches wide and 0.25 inch thick running at a speed of 60 feet per second. The tension in the slack side of the belt is 0.45 of that in the tight side, and the maximum allowable stress per square inch of belt section is 280 pounds. Taking the weight of a cubic inch of leather as 0.036 pound, to what extent does the effect of centrifugal action reduce the power transmitted?

Ans. 20.87; 17.2 per cent.

19. It is required to transmit 16 H.P. from a pulley 20 inches in diameter by means of a belt which embraces only two-ninths of the circumference of the pulley. The thickness of the belt is three-eighths of an inch, the safe working stress is 300 pounds per square inch of belt section, and the pulley speed is 120 revolutions per minute. Find the tensions in the two parts of the belt, and the width of belt required.

Ans. 2174.2 pounds; 1333.8 pounds; 19.32 inches.

20. What horse power will be transmitted from a 10-foot pulley by 12 ropes of 1.5 inch diameter, the revolutions being 140 per minute, the coefficient of friction 0.14, and the arc of contact 130 degrees? *Ans.* 306.

21. A drive of 20 ropes transmits 600 H.P. from a 10-foot pulley making 100 revolutions per minute, the arc of contact of the smaller pulley being 160 degrees and the coefficient of friction 0.12. What is the pull in each rope? What should be the diameter of the ropes?

Ans. 364 pounds; 1.75 inches.

CHAPTER II

TRANSMISSION BY TOOTHED WHEELS

19. Toothed Wheel Gearing. — The usual arrangement of toothed wheels in train for the transmission of power is to have two wheels of unequal size on each shaft except the first and last, making the smaller wheel of a pair on one shaft gear with the larger of the pair on the next shaft in the series. The two wheels of unequal size on a shaft is the mechanical equivalent of a lever with unequal arms, and therefore modifies the power that may be transmitted.

The circles of Fig. 8 represent the pitch circles of two toothed, or spur, wheels in gear. The pitch circles of two wheels in gear are circles which appear to roll upon each other and which pass, approximately, through the middle of the elevation of the teeth. They may be regarded as the outlines of two discs which roll together by the friction at the circumferences.

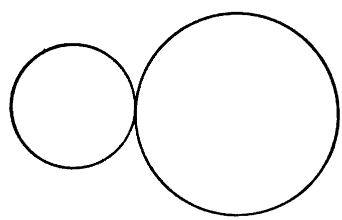


Fig. 8.

There can be no slipping of one pitch circle over the other, owing to the teeth; therefore the same length of circumference of each must pass over the point of contact in a given time. If D and d represent the diameters of the large and small wheels respectively, and N and n the number of their revolutions per unit of time, then

$$\pi DN = \pi dn, \quad \text{or} \quad DN = dn.$$

The teeth on the two wheels are the same size, and their number will be proportional to the diameters of the wheels, and we may write $AN = Bn$, in which A and B are the numbers of teeth of the large and small wheels respectively.

The pitch of the teeth is the distance measured on the circumference of the pitch circle between the centers of two consecutive teeth. The radius of the pitch circle of a spur wheel may be found by dividing the product of the pitch and number of teeth by 2π.

The formulas for belt pulleys hold for toothed wheels, for the belt performs the same office as the teeth — it causes the circumference of each pulley to move over the same distance in the same time.

With toothed gearing the slipping of belt gearing is avoided and an exact velocity ratio may be maintained, provided the teeth are carefully constructed on certain geometrical principles.

20. **Arrangement of a Train of Toothed Wheels.** — An arrangement of a train of toothed wheels for the transmission of power is shown in Fig. 9, where two wheels of unequal size are placed on each axis, except the first and last, and where the smaller wheel of any pair gears with the larger wheel of the next pair in the train. On the first and last axis there is but one wheel.

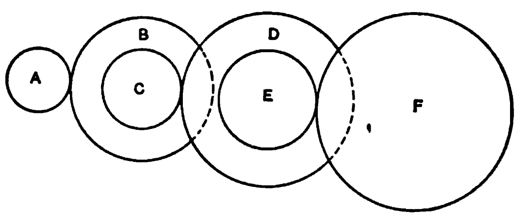


Fig. 9.

In this arrangement for power transmission the driving wheels are smaller than the followers, the revolutions of the successive

axles decreasing in number. Should a high velocity ratio be the object desired, the arrangement would be the reverse of that for power transmission, the driving wheels being the larger, and, in consequence, the revolutions of the successive axles would increase in number.

Suppose the power to be applied at wheel A. Then wheels A, C, and E will be the drivers and wheels B, D, and F the followers. Let the letters denoting the wheels denote also the numbers of teeth in the wheels respectively. Let N_1, N_2, N_3 denote the number of revolutions in a period of time of the drivers A, C, E respectively; and let n_1, n_2, n_3 be like representations of the followers B, D, F.

From what has been shown we shall have

$$AN_1 = Bn_1, \quad CN_2 = Dn_2, \quad \text{and} \quad EN_3 = Fn_3.$$

Multiplying these equations, member by member, we have

$$AN_1 \times CN_2 \times EN_3 = Bn_1 \times Dn_2 \times Fn_3.$$

Since B and C are fixed on the same shaft, we have $N_2 = n_1$; and for the same reason $N_3 = n_2$.

Hence $AN_1 \times C \times E = B \times D \times Fn_3$, whence $\dfrac{A \times C \times E}{B \times D \times F} = \dfrac{n_3}{N_1}$.

That is, the product of the number of teeth in all the drivers, divided by the product of the number of teeth in all the followers, is equal to the ratio of the number of revolutions of the last wheel to the number of revolutions of the first wheel. This ratio is the *value* of the train, and is denoted by e.

If, in Fig. 9, $A = 18$, $C = 20$, $E = 24$, $B = 36$, $D = 40$, $F = 72$, and $N_1 = 120$, we shall have

$$e = \frac{18 \times 20 \times 24}{36 \times 40 \times 72} = \frac{n_3}{120}, \quad \text{whence } n_3 = 10,$$

that is, the last wheel will make 10 revolutions while the first wheel is making 120 revolutions.

When any number of wheels are in gear, no two of them being on the same axis, as in Fig. 10, the combination is the equivalent only of a single pair of wheels, viz., the first wheel and the last wheel; the intervening wheels simply transfer the motion and determine the direction of rotation of the last wheel. If the number of idler wheels intervening between the first and last wheel be odd, the direction of rotation of the first and last wheels will be the same; if even, the rotations will be in opposite directions.

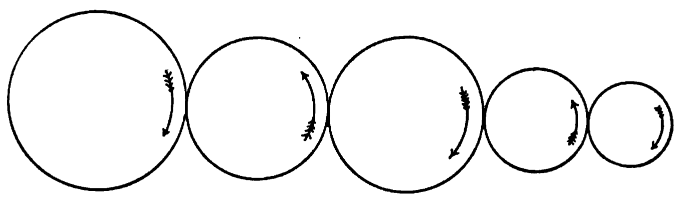


Fig. 10.

21. Driving End and Load End of Gearing. — Generally speaking, the part of a system of gearing to which the motive power is applied is called the *driving end*, and the part at which the resistance is overcome, or at which the useful work is done, is called the *load end*. This applies to the operation of all machines, and in general terms we have these relations:

$$\text{Velocity ratio} = \frac{\text{Movement of driving end}}{\text{Movement of load end}},$$

$$\text{Mechanical advantage} = \frac{\text{Load}}{\text{Driving force}},$$

$$\text{Mechanical efficiency} = \frac{\text{Useful work performed}}{\text{Total work expended}}.$$

22. Toothed Gearing of Screw-cutting Lathes. — The toothed gearing of a screw-cutting lathe is a very important application of wheels in train. The driving wheel of the gearing is either fast on the lathe spindle or derives motion by means of inter-

mediate gearing. In either case the revolutions of the first driver are the same as those of the lathe spindle. The intermediate gearing affords a ready means of throwing the gear wheels out of action when the lathe is to be run at high speeds, as for polishing; it also enables the direction of rotation of the lead screw of the lathe to be changed at will. If, as is usually the case, the lead screw is right-handed, the screw to be cut will be right-handed or left-handed, according as the direction of rotation of the lead screw is the same as, or different from, that of the lathe spindle.

There are two systems of lathe gearing,—*simple* and *compound.*

23. **Simple Lathe Gearing.** — Fig. 11 is a representation of simple lathe gearing, no two wheels being on the same axis. Any wheels intervening between the driving wheel A and the wheel C on the lead screw of the lathe have no influence other than to convey the motion, so that the only wheels to be considered are the driver A and the wheel C on the lead screw. If the gearing be such that the lathe spindle and the lead screw make the same number of turns, the thread cut will have the same pitch as the thread on the lead screw. In all other cases the pitch of the thread to be cut will be finer or coarser than that on the lead screw in the exact proportion that the revolutions of the lathe spindle in a unit of time are greater or less than those of the lead screw.

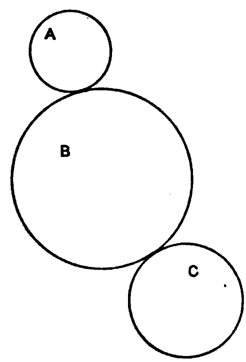


Fig. 11.

The driving wheel A is fast on the lathe spindle, or so connected with it as to make the same number of revolutions as the spindle. The wheel C is keyed on the lead screw of the lathe and the intermediate wheel B is an idler, serving the purpose of making the wheels A and C rotate in the same

direction and also of filling in the space between A and C, the axis of B being adjustable in a slotted arm.

The lead screw of the lathe works in a nut on the lathe carriage which carries the cutting tool, so that for each revolution of the wheel C the cutting tool advances a distance equal to the pitch of the lead screw, and it depends entirely upon the ratio between the numbers of teeth of wheels C and A as to the number of turns the piece of work upon which the thread is to be cut will make while the cutting tool is moving through that distance. From this we shall have

$$e = \frac{\text{Revolutions of lead screw}}{\text{Revolutions of lathe spindle}} = \frac{\text{Pitch of screw to be cut}}{\text{Pitch of lead screw}}.$$

The problem of screw-cutting consists, then, in finding a train of wheels in which we shall have

$$e = \frac{\text{Pitch of screw to be cut}}{\text{Pitch of lead screw}}.$$

Example. — It is desired to cut a screw of 10 threads to the inch with a lathe whose lead screw has a pitch of $\frac{1}{4}$ inch. Find a suitable train of wheels.

Solution. —

$$e = \frac{\frac{1}{10}}{\frac{1}{4}} = \frac{2}{5} = \frac{20}{50}.$$

Giving 20 teeth to the wheel A and 50 teeth to wheel C is one of a number of solutions.

24. Compound Lathe Gearing. — Fig. 12 is a representation of compound lathe gearing. The two wheels, B and C, of different size on the axis intervening between the driver A and the wheel D on the lead screw are factors in the value of the train. From Art. 20 we have

$$e = \frac{A \times C}{B \times D}.$$

Example. — It is desired to cut a screw of 16 threads to the inch with a lathe whose lead screw has a pitch of $\frac{1}{2}$ inch. Find a suitable train of compound gearing.

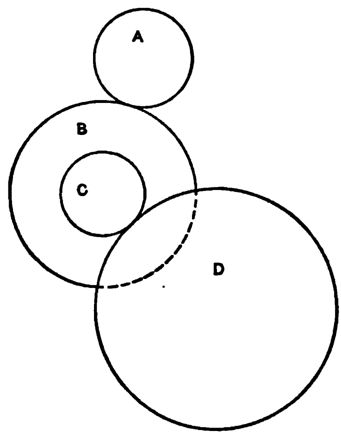


Fig. 12.

Solution.— $e = \frac{\frac{1}{16}}{\frac{1}{2}} = \frac{1}{8} = \frac{1}{2} \times \frac{1}{4} = \frac{20}{40} \times \frac{24}{96} = \frac{A \times C}{B \times D}.$

Hence, referring to Fig. 12, giving 20 teeth to A, and 24, 40, and 66 to C, B, and D, respectively, is a probable solution.

In order to cut a left-hand thread the lathe spindle and the lead screw must turn in opposite directions. This may be effected by interposing an idle wheel between C and D.

25. **Back Gear Lathe Attachment.** — The back gear attachment to a lathe is a mechanical arrangement to increase the power of the lathe at the expense of the speed. The cone pulley to which the pinion A is attached, Fig. 13, is loose on the lathe spindle. The spur wheel B is keyed to the spindle. The spur wheel C and pinion D, carried on the shaft E, form the back gear and can, at will, be thrown into or out of gear with A and B.

The motion of the cone pulley may be conveyed to the lathe spindle in two ways: (*a*) The back gear being disengaged, as shown in Fig. 13, the spur wheel *B* is made to engage with the speed cone by means of a bolt, thereby giving to the lathe spindle the same revolutions that are made by the cone pulley. (*b*) Disengaging the wheel *B* from the speed cone and throwing the back gear into gear with wheels *A* and *B*, the motion of the cone pulley is then transmitted to the lathe spindle by means of the train *ACDB*, making the entire system consist of a train of which the pulley on the countershaft is the first driver and the spur wheel *B* the last follower.

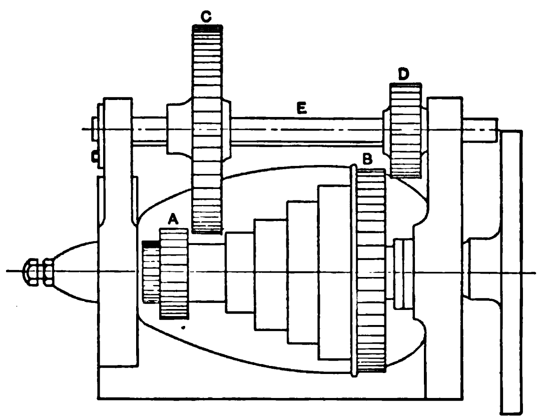


Fig. 13.

26. Transmission of Power by Toothed Wheels. — In the different mechanisms employing a train of toothed wheels for the transmission of power, either the movement of a small power through a comparatively great distance is utilized in overcoming a much greater resistance through a much smaller distance; or, conversely, the movement of a large power through a small distance is utilized in making a smaller resistance move through a much greater distance. In the two cases the desired results were power and velocity respectively, the underlying mechanical

principle being that, what is gained in power is lost in speed, and conversely.

The power may be applied by hand to the end of a lever, or to the crank pin of an engine, the lever or crank to be rigidly connected to the first axis of the train.

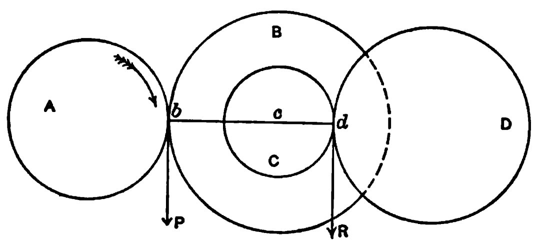


Fig. 14.

Suppose the wheels B and C of Fig. 14 to be fixed on the axis c. The driving power P is applied through the wheel A tangentially to the pitch surface of the teeth in contact at b, the tendency being to turn the wheels B and C in a contra-clockwise direction about axis c. The action of P is resisted by the reaction R of the load applied tangentially to the pitch surfaces of the teeth of the wheels C and D in contact at d, with a tendency to turn the wheels B and C in a clockwise direction about the axis c. With these opposite turning tendencies we shall have, when motion is about to take place,

$$P \times cb = R \times cd,$$

whence

$$\frac{P}{R} = \frac{cd}{cb} = \frac{\text{Radius of wheel } C}{\text{Radius of wheel } B} = \frac{\text{Number of teeth in wheel } C}{\text{Number of teeth in wheel } B}.$$

Example. — The double purchase wheelwork of Fig. 15 represents a common arrangement applied to hoisting machinery, such as cranes. The numbers attached to the spur wheels and pinions indicate the number of teeth they contain. The length

of the lever handles being 18 inches, the radius of the drum 10 inches, and the power applied to each of the handles 40 pounds, it is required to find: (*a*) The weight raised at the drum; (*b*) the tangential pressures between the teeth of the wheels; (*c*) the horse power transmitted, supposing the handles to make 24 turns per minute.

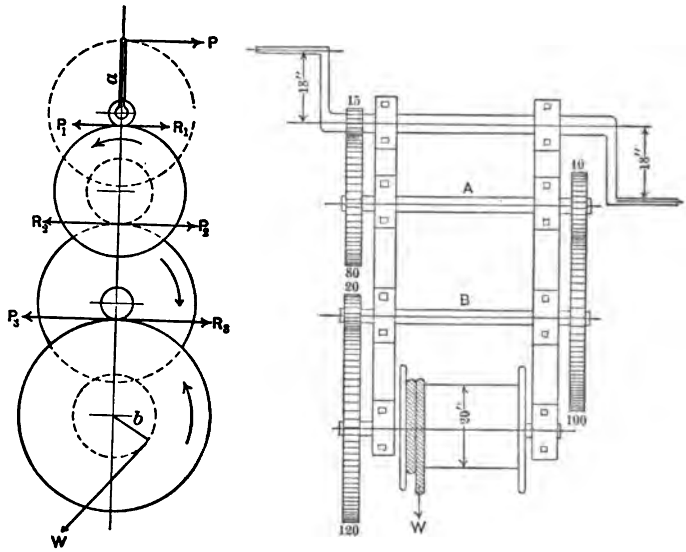


Fig. 15.

Solution. — Since the radii of wheels are proportional to the number of teeth, we may denote by $15x$ and $120x$ the radii of the first pinion and the last spur wheel of the train respectively.

Denoting the length of the power handles and the radius of the drum by a and b respectively, we have, by the principle of moments, these equations:

$$P \times a = R_1 \times 15x \quad (1)$$

$$P_1 \times 80 = R_2 \times 40 \quad (2)$$

$$P_2 \times 100 = R_3 \times 20 \quad (3)$$

$$P_3 \times 120x = W \times b \quad (4)$$

Multiplying these equations, member by member, remembering that $P_1 = R_1$, $P_2 = R_2$, and $P_3 = R_3$, we have

$$Pa \times 80 \times 100 \times 120 = 15 \times 40 \times 20 \times Wb,$$

whence

$$\frac{P}{W} = \frac{15 \times 40 \times 20}{80 \times 100 \times 120} \cdot \frac{b}{a} = e \cdot \frac{b}{a} = \frac{1}{80} \times \frac{10}{18} = \frac{1}{144},$$

and $$W = 11{,}520 \text{ pounds.}$$

In the above calculations the effect of friction and of the diameter of the rope have been neglected. Friction would likely reduce the result as much as 30 per cent, and the effective drum radius would be the radius of the drum plus the radius of the rope.

The tangential pressures between the surfaces of the teeth in contact may be found by means of equations (1), (2), (3), and (4) if the pitch of the teeth be known.

Suppose the pitch of the teeth to be 1.25 inches. Then

$$\text{Radius of first pinion} = \frac{1.25 \times 15}{2\pi} = \frac{75}{8\pi}.$$

From equation (1) we have

$$80 \times 18 = R_1 \times \frac{75}{8\pi}, \quad \text{whence } R_1 = 482.5 \text{ pounds.}$$

From equation (2) we have

$$482.5 \times 80 = R_2 \times 40, \quad \text{whence } R_2 = 965 \text{ pounds.}$$

From equation (3) we have

$$965 \times 100 = R_3 \times 20, \quad \text{whence } R_3 = 4825 \text{ pounds.}$$

These results indicate that the teeth of the wheels should be made stronger as the drum shaft is approached. This could be done by using a coarser pitch for the last pair of wheels in gear.

If Q denotes the tangential pressure at the pitch surface of

any wheel in the train, and V the velocity in feet per minute of the wheel at the pitch surface, then

$$\text{H.P. transmitted} = \frac{QV}{33{,}000},$$

and this is constant for any stage of the transmission.

Thus, if the lever handles make 24 turns per minute we shall have at the lever handle axle

$$\text{H.P.} = \frac{QV}{33{,}000} = \frac{80 \times 2\pi \times 18 \times 24}{33{,}000 \times 12} = 0.548.$$

Since the value of the train is $\frac{1}{80}$, the last wheel will make $\frac{24}{80} = \frac{3}{10}$ of a revolution in a minute, and the radius of the last wheel is $\frac{1.25 \times 120}{2\pi} = \frac{75}{\pi}$.

Hence at the drum axle

$$\text{H.P.} = \frac{4825 \times 2\pi \times 75 \times 3}{33{,}000 \times 12 \times \pi \times 10} = 0.548.$$

In like manner, the horse power transmitted from axle A to axle B may be shown to be the same, thus:

Axle A makes $\frac{15 \times 24}{80} = 4.5$ revolutions per minute, and the radius of the pinion of 40 teeth is $\frac{1.25 \times 40}{2\pi} = \frac{25}{\pi}$. Then

$$\text{H.P. from axle } A \text{ to axle } B = \frac{965 \times 2\pi \times 25 \times 4.5}{33{,}000 \times \pi \times 12} = 0.548.$$

In illustration of a combination of belt and toothed wheel gears this example is given:

The motion of an engine shaft is communicated to a 15-inch pulley on a shaft A by means of an open belt passing from a 36-inch flywheel. The motion of shaft A is transmitted to a shaft B by means of spur wheels of $\frac{7}{8}$-inch pitch and of 43 and 24 teeth, an idle wheel intervening. The motion of shaft B is communicated to the spindle of a fan by means of an open

belt passing over an 18-inch pulley on shaft B to a 12-inch pulley on the fan spindle. The engine makes 250 revolutions per minute and has a stroke of 12 inches; the mean pressure on the crank pin is 90 pounds, and the ratio of the tensions in the tight and slack sides of the flywheel belt is 2.25 : 1. Find: (*a*) The number of revolutions per minute of the fan; (*b*) the tangential pressure at the point of contact of the teeth in gear; (*c*) the horse power at two or more stages of the transmission; (*d*) the difference of the tensions in the tight and slack sides of the fan belt.

Solution.—

$$\text{Revolutions of fan per minute} = \frac{250 \times 36 \times 43 \times 18}{15 \times 24 \times 12} = 1612.5.$$

Taking moments about the axis of the engine shaft, Fig. 16, we have

$$18\,T_1 = 90 \times 6 + 18\,T_2, \quad \text{whence} \quad T_1 - T_2 = 30.$$

Then $$2.25\,T_2 - T_2 = 30, \quad \text{whence} \quad T_2 = 24,$$

and $$T_1 = 30 + 24 = 54 \text{ pounds.}$$

$$\text{Radius of spur wheel of 43 teeth} = \frac{7 \times 43}{8 \times 2\pi} = 5.95, \text{ say 6 inches.}$$

$$\text{Revolutions of shaft } A = \frac{250 \times 36}{15} = 600 \text{ per minute.}$$

To find the tangential pressure, P, at the point of contact, a, of the teeth in gear, we take moments about the axis of shaft A, thus,

$$7.5\,T_1 = 6\,P + 7.5\,T_2,$$

whence $$P = \frac{7.5\,(T_1 - T_2)}{6} = \frac{7.5 \times 30}{6} = 37.5 \text{ lbs.},$$

and this pressure is transmitted without change to the point of contact, b, the intervening spur wheel having no effect other

than to cause the fan to turn in the same direction as the engine.

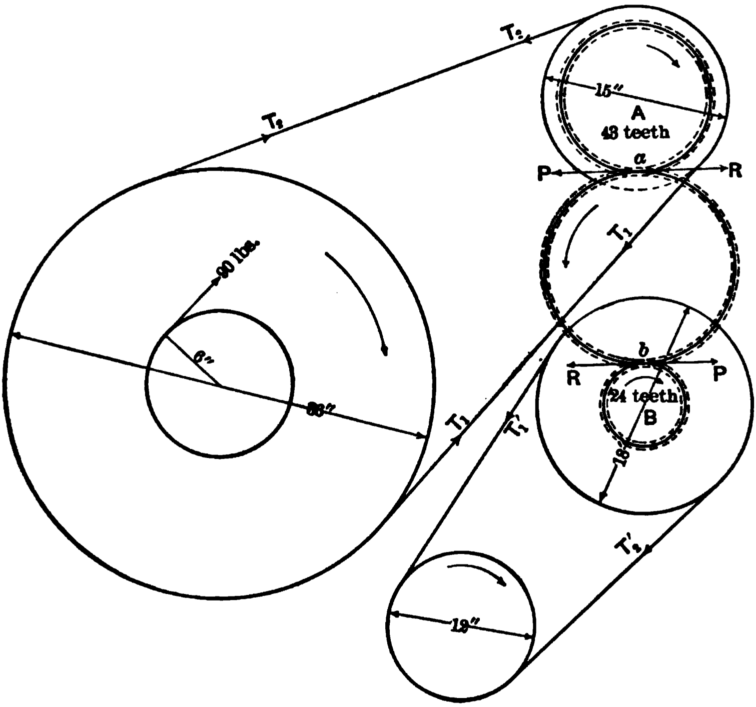


Fig. 16.

$$\text{H.P. at engine shaft} = \frac{PV}{33{,}000} = \frac{90 \times 12\pi \times 250}{33{,}000 \times 12} = 2.142.$$

$$\text{H.P. at shaft } A = \frac{(T_1 - T_2)V}{33{,}000} = \frac{30 \times 15\pi \times 600}{33{,}000 \times 12} = 2.142,$$

when considering the 15-inch pulley on shaft A.

The horse power being the same at all points in the transmission, we shall have, when considering the 18-inch pulley on shaft B and the tensions in the fan belt,

$$\text{H.P.} = 2.142 = \frac{(T_1' - T_2')\, 18\pi \times 600 \times 43}{33{,}000 \times 12 \times 24},$$

whence $$T_1' - T_2' = 13.9 \text{ pounds.}$$

This last result may be obtained by a consideration of the equilibrium of the whole system, thus:

The resistance R between the teeth in contact at a maintains with T_1 and T_2 the equilibrium about the axis of shaft A. The contraclockwise moment of P at a is balanced by the clockwise moment of R at b, while P at b maintains with T_1' and T_2' the equilibrium about the axis of shaft B.

The radius of the wheel of 24 teeth on shaft B being $\dfrac{7 \times 24}{8 \times 2\pi}$ $= 3.34$ inches, we shall have, by moments about the axis of shaft B,

$$9\,T_1' = 3.34\,P + 9\,T_2',$$

or
$$9\,(T_1' - T_2') = 3.34 \times 37.5,$$

whence
$$T_1' - T_2' = 13.9 \text{ pounds}.$$

as found above.

These calculations neglect the losses in the transmission due to belt slipping and friction between the teeth and at the bearings, the aggregate of which would probably exceed 30 per cent.

PROBLEMS

1. A, B, and C are three parallel spindles, A carrying a spur wheel of 52 teeth which gears with one of 19 teeth on B. On B is another wheel of 81 teeth gearing with one of 21 teeth on C. While A is making 15 turns, how many will C make? How many will B make?

Ans. 158.33; 41.05.

2. If the change wheels of a lathe have 18, 30, 40, 50, and 88 teeth, show an arrangement for cutting a screw of 11 threads to the inch, the lead screw having a pitch of one-third inch.

3. With a lead screw of one-half inch pitch, and change wheels of 20, 24, 30, 40, 55, 60, 80, and 100, show a selection of wheels to cut screws of 6, 11, and 16 threads to the inch.

4. The back gear of a lathe is in use. Diameter of pulley on counter shaft, 4.25 inches; diameter of pulley of lathe, 8.75 inches; pinion on cone pulley of lathe, 18 teeth; spur wheel on back shaft, 58 teeth; pinion on back shaft, 18 teeth; spur wheel on lathe spindle, 58 teeth. How many revolutions per minute will the lathe spindle make when the countershaft is making 150 revolutions per minute? *Ans.* 7.017.

5. Two parallel shafts, whose axes are to be, as nearly as possible, 30 inches apart, are to be connected by a pair of spur wheels, so that while the driver runs at 100 revolutions per minute the follower is required to run at only 25 revolutions per minute. Sketch the arrangement, and mark on each wheel its pitch diameter and the number of its teeth, the pitch of the teeth being 1.25 inches. Determine also the exact distance apart of the two shafts. *Ans.* 29.8295 inches.

6. Make a sketch of a back gear of a lathe. If the two wheels have 63 teeth each, and each pinion 25 teeth, find the reduction in the velocity ratio of the lathe due to the back gear. *Ans.* 1 : 6.35.

7. The double purchase wheelwork of a crane consists of a pinion of 16 teeth on the handle axle; a wheel and pinion of 64 and 20 teeth respectively on the first intermediate axle; a wheel and pinion of 80 and 18 teeth respectively on the second intermediate axle, and a wheel of 90 teeth on the drum axle. The power handles are 20 inches long, the radius of the drum 11 inches, the diameter of the rope 2 inches, and the pitch of the teeth 1.25 inches. Neglecting friction, it is required to find: (1) The power that must be applied at the handles to raise 9600 pounds at the drum; (2) the tangential pressures between the teeth of the wheels in gear; (3) the H.P. transmitted, supposing the handles to make 20 turns per minute.

Ans. 72 lbs.; 452.4 lbs.; 1447.7 lbs.; 6434 lbs.; 0.4569 H.P.

8. The wheelwork of a crane consists of a pinion of 11 teeth gearing with a wheel of 92 teeth, and of a pinion of 12 teeth gearing with a wheel of 72 teeth on the drum axle. The lever handle being 18 inches long, and the radius of the drum 9 inches, it is required to find the ratio of the power to the weight raised. *Ans.* 1 : 100 nearly.

9. In the example of a combination of belt and toothed wheel gears given at the end of Chapter II, it is required to find the horse power of the transmission at shafts *A* and *B* by considering the spur wheel of 43 teeth on shaft *A* and the one of 24 teeth on shaft *B*.

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